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An introduction to homological algebra

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To my wife, Laurel Van Leer, whose support is invaluable,
and to my children, Chad and Aubrey, without whom
this book would have been completed much sooner.

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Introduction

Homological algebra is a tool used to prove nonconstructive existence theorems in algebra (and in algebraic topology). It also provides obstructions to carrying out various kinds of constructions; when the obstructions are zero, the construction is possible. Finally, it is detailed enough so that actual calculations may be performed in important cases. The following simple question (taken from Chapter 3) illustrates these points: Given a subgroup A of an abelian group B and an integer n, when is nA the intersection of A and nB? Since the cyclic group $\mathbb{Z}/n$ is not flat, this is not always the case. The obstruction is the group $\mathrm{Tor}(B/A, \mathbb{Z}/n)$, which explicitly is $\{x \in B/A : nx = 0\}$.

This book intends to paint a portrait of the landscape of homological algebra in broad brushstrokes. In addition to the "canons" of the subject (Ext, Tor, cohomology of groups, and spectral sequences), the reader will find introductions to several other subjects: sheaves, $\lim^1$, local cohomology, hypercohomology, profinite groups, the classifying space of a group, Affine Lie algebras, the Dold-Kan correspondence with simplicial modules, triple cohomology, Hochschild and cyclic homology, and the derived category. The historical connections with topology, regular local rings, and semisimple Lie algebras are also described.

After a lengthy gestation period (1890–1940), the birth of homological algebra might be said to have taken place at the beginning of World War II with the crystallization of the notions of homology and cohomology of a topological space. As people (primarily Eilenberg) realized that the same formalism could be applied to algebraic systems, the subject exploded outward, touching almost every area of algebra. This phase of development reached maturity in 1956 with the publication of Cartan and Eilenberg's book [CE] and with the emergence of the central notions of derived functors, projective modules, and injective modules.

Until 1970, almost every mathematician learned the subject from Cartan-Eilenberg [CE]. The canonical list of subjects (Ext, Tor, etc.) came from this book. As the subject gained in popularity, other books gradually appeared on the subject: MacLane's 1963 book [MacH], Hilton and Stammbach's 1971 book [HS], Rotman's 1970 notes, later expanded into the book [Rot], and Bourbaki's 1980 monograph [BX] come to mind. All these books covered the canonical list of subjects, but each had its own special emphasis.

In the meantime, homological algebra continued to evolve. In the period 1955–1975, the subject received another major impetus, borrowing topological ideas. The Dold-Kan correspondence allowed the introduction of simplicial methods, $\lim^1$ appeared in the cohomology of classifying spaces, spectral sequences assumed a central role in calculations, sheaf cohomology became part of the foundations of algebraic geometry, and the derived category emerged as the formal analogue of the topologists' homotopy category.

Largely due to the influence of Grothendieck, homological algebra became increasingly dependent on the central notions of abelian category and derived functor. The cohomology of sheaves, the Grothendieck spectral sequence, local cohomology, and the derived category all owe their existence to these notions. Other topics, such as Galois cohomology, were profoundly influenced.

Unfortunately, many of these later developments are not easily found by students needing homological algebra as a tool. The effect is a technological barrier between casual users and experts at homological algebra. This book is an attempt to break down that barrier by providing an introduction to homological algebra as it exists today.

This book is aimed at a second- or third-year graduate student. Based on the notes from a course I taught at Rutgers University in 1985, parts of it were used in 1990–92 in courses taught at Rutgers and Queens' University (the latter by L. Roberts). After Chapter 2, the teacher may pick and choose topics according to interest and time constraints (as was done in the above courses).

As prerequisites, I have assumed only an introductory graduate algebra course, based on a text such as Jacobson's *Basic Algebra* I [BAI]. This means some familiarity with the basic notions of category theory (category, functor, natural transformation), a working knowledge of the category **Ab** of abelian groups, and some familiarity with the category *R*–**mod** (resp. **mod**–*R*) of left (resp. right) modules over an associative ring *R*. The notions of abelian category (section 1.2), adjoint functor (section 2.3) and limits (section 2.6) are introduced in the text as they arise, and all the category theory introduced in this book is summarized in the Appendix. Several of the motivating examples assume an introductory graduate course in algebraic topology but may be skipped over by the reader willing to accept that such a motivation exists. An exception is the last section (section 10.9), which requires some familiarity with point-set topology.

Many of the modern applications of homological algebra are to algebraic geometry. Inasmuch as I have not assumed any familiarity with schemes or algebraic geometry, the reader will find a discussion of sheaves of abelian groups, but no mention of sheaves of $\mathcal{O}_X$-modules. To include it would have destroyed the flow of the subject; the interested reader may find this material in [Hart].

Chapter 1 introduces chain complexes and the basic operations one can make on them. We follow the indexing and sign conventions of Bourbaki [BX], except that we introduce two total complexes for a double complex: the algebraists' direct sum total complex and the topologists' product total complex. We also generalize complexes to abelian categories in order to facilitate the presentation of Chapter 2, and also in order to accommodate chain complexes of sheaves.

Chapter 2 introduces derived functors via projective modules, injective modules, and δ-functors, following [Tohoku]. In addition to Tor and Ext, this allows us to define sheaf cohomology (section 2.5). Our use of the acyclic assembly lemma in section 2.7 to balance Tor and Ext is new.

Chapter 3 covers the canonical material on Tor and Ext. In addition, we discuss the derived functor $\lim^1$ of the inverse limit of modules (section 3.5), the Künneth Formulas (section 3.6), and their applications to algebraic topology.

Chapter 4 covers the basic homological developments in ring theory. Our discussion of global dimension (leading to commutative regular local rings) follows [KapCR] and [Rot]. Our material on Koszul complexes follows [BX], and of course the material on local cohomology is distilled from [GLC].

Spectral sequences are introduced in Chapter 5, early enough to be able to utilize this fundamental tool in the rest of the book. (A common problem with learning homological algebra from other textbooks is that spectral sequences are often ignored until the last chapter and so are not used in the textbook itself.) Our basic construction follows [CE]. The motivational section 5.3 on the Leray-Serre spectral sequence in topology follows [MacH] very closely. (I first learned about spectral sequences from discussions with MacLane and this section of his book.) Our discussion of convergence covers several results not in the standard literature but widely used by topologists, and is based on unpublished notes of M. Boardman.

In Chapter 6 we finally get around to the homology and cohomology of groups. The material in this chapter is taken from [Brown], [MacH], and [Rot].

We use the Lyndon/Hochschild-Serre spectral sequence to do calculations in section 6.8, and introduce the classifying space BG in section 6.10. The material on universal central extensions (section 6.9) is based on [Milnor] and [Suz]. The material on Galois cohomology (and the Brauer group) comes from [BAII], [Serre], and [Shatz].

Chapter 7 concerns the homology and cohomology of Lie algebras. As Lie algebras aren't part of our prerequisites, the first few sections review the subject, following [JLA] and [Humph]. Most of our material comes from the 1948 Chevalley-Eilenberg paper [ChE] and from [CE], although the emphasis, and our discussion of universal central extensions and Affine Lie algebras, comes from discussions with R. Wilson and [Wil].

Chapter 8 introduces simplicial methods, which have long been a vital part of the homology toolkit of algebraic topologists. The key result is the Dold-Kan theorem, which identifies simplicial modules and positive chain complexes of modules. Applied to adjoint functors, simplicial methods give rise to a host of canonical resolutions (section 8.6), such as the bar resolution, the Godement resolution of a sheaf [Gode], and the triple cohomology resolutions [BB]. Our discussion in section 8.7 of relative Tor and Ext groups parallels that of [MacH], and our short foray into André-Quillen homology comes from [Q] and [Barr].

Chapter 9 discusses Hochschild and cyclic homology of k-algebras. Although part of the discussion is ancient and is taken from [MacH], most is new. The material on differentials and smooth algebras comes from [EGA, IV] and [Mat]. The development of cyclic homology is rather new, and textbooks on it ([Loday],[HK]) are just now appearing. Much of this material is based on the articles [LQ], [Connes], and [Gw].

Chapter 10 is devoted to the derived category of an abelian category. The development here is based upon [Verd] and [HartRD]. The material on the topologists' stable homotopy in section 10.9 is based on [A] and [LMS].

Paris, February 1993

Contents

1
Chain Complexes

1.1 Complexes of R-Modules

Homological algebra is a tool used in several branches of mathematics: algebraic topology, group theory, commutative ring theory, and algebraic geometry come to mind. It arose in the late 1800s in the following manner. Let f and g be matrices whose product is zero. If $g \cdot v = 0$ for some column vector v, say, of length n, we cannot always write $v = f \cdot u$. This failure is measured by the *defect*

$$d = n - \text{rank}(f) - \text{rank}(g).$$

In modern language, f and g represent linear maps

$$U \xrightarrow{f} V \xrightarrow{g} W$$

with $gf = 0$, and d is the dimension of the *homology module*

$$H = \ker(g)/f(U).$$

In the first part of this century, Poincaré and other algebraic topologists utilized these concepts in their attempts to describe "n-dimensional holes" in simplicial complexes. Gradually people noticed that "vector space" could be replaced by "R-module" for any ring R.

This being said, we fix an associative ring R and begin again in the category **mod**–R of right R-modules. Given an R-module homomorphism $f: A \to B$, one is immediately led to study the kernel $\ker(f)$, cokernel $\text{coker}(f)$, and image $\text{im}(f)$ of f. Given another map $g: B \to C$, we can form the sequence

$$(*) \qquad A \xrightarrow{f} B \xrightarrow{g} C.$$

We say that such a sequence is *exact* (at B) if $\ker(g) = \operatorname{im}(f)$. This implies in particular that the composite $gf: A \to C$ is zero, and finally brings our attention to sequences $(*)$ such that $gf = 0$.

Definition 1.1.1 A *chain complex* $C_.$ of R-modules is a family $\{C_n\}_{n\in\mathbb{Z}}$ of R-modules, together with R-module maps $d = d_n: C_n \to C_{n-1}$ such that each composite $d \circ d: C_n \to C_{n-2}$ is zero. The maps d_n are called the *differentials* of $C_.$. The kernel of d_n is the module of *n-cycles* of $C_.$, denoted $Z_n = Z_n(C_.)$. The image of $d_{n+1}: C_{n+1} \to C_n$ is the module of *n-boundaries* of $C_.$, denoted $B_n = B_n(C_.)$. Because $d \circ d = 0$, we have

$$0 \subseteq B_n \subseteq Z_n \subseteq C_n$$

for all n. The n^{th} *homology module* of $C_.$ is the subquotient $H_n(C_.) = Z_n/B_n$ of C_n. Because the dot in $C_.$ is annoying, we will often write C for $C_.$.

Exercise 1.1.1 Set $C_n = \mathbb{Z}/8$ for $n \geq 0$ and $C_n = 0$ for $n < 0$; for $n > 0$ let d_n send $x \pmod 8$ to $4x \pmod 8$. Show that $C_.$ is a chain complex of $\mathbb{Z}/8$–modules and compute its homology modules.

There is a category **Ch(mod–**R**)** of chain complexes of (right) R-modules. The objects are, of course, chain complexes. A *morphism* $u: C_. \to D_.$ is a chain complex map, that is, a family of R-module homomorphisms $u_n: C_n \to D_n$ commuting with d in the sense that $u_{n-1}d_n = d_{n-1}u_n$. That is, such that the following diagram commutes

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d} & C_{n+1} & \xrightarrow{d} & C_n & \xrightarrow{d} & C_{n-1} & \xrightarrow{d} & \cdots \\
 & & \downarrow u & & \downarrow u & & \downarrow u & & \\
\cdots & \xrightarrow{d} & D_{n+1} & \xrightarrow{d} & D_n & \xrightarrow{d} & D_{n-1} & \xrightarrow{d} & \cdots .
\end{array}$$

Exercise 1.1.2 Show that a morphism $u: C_. \to D_.$ of chain complexes sends boundaries to boundaries and cycles to cycles, hence maps $H_n(C_.) \to H_n(D_.)$. Prove that each H_n is a functor from **Ch(mod–**R**)** to **mod–**R.

Exercise 1.1.3 (Split exact sequences of vector spaces) Choose vector spaces $\{B_n, H_n\}_{n\in\mathbb{Z}}$ over a field, and set $C_n = B_n \oplus H_n \oplus B_{n-1}$. Show that the projection-inclusions $C_n \to B_{n-1} \subset C_{n-1}$ make $\{C_n\}$ into a chain complex, and that every chain complex of vector spaces is isomorphic to a complex of this form.

Exercise 1.1.4 Show that $\{\mathrm{Hom}_R(A, C_n)\}$ forms a chain complex of abelian groups for every R-module A and every R-module chain complex C. Taking $A = Z_n$, show that if $H_n(\mathrm{Hom}_R(Z_n, C)) = 0$, then $H_n(C) = 0$. Is the converse true?

Definition 1.1.2 A morphism $C_. \to D_.$ of chain complexes is called a *quasi-isomorphism* (Bourbaki uses *homologism*) if the maps $H_n(C_.) \to H_n(D_.)$ are all isomorphisms.

Exercise 1.1.5 Show that the following are equivalent for every $C_.$:

1. $C_.$ is *exact*, that is, exact at every C_n.
2. $C_.$ is *acyclic*, that is, $H_n(C_.) = 0$ for all n.
3. The map $0 \to C_.$ is a quasi-isomorphism, where "0" is the complex of zero modules and zero maps.

The following variant notation is obtained by reindexing with superscripts: $C^n = C_{-n}$. A *cochain complex* $C^\cdot$ of R-modules is a family $\{C^n\}$ of R-modules, together with maps $d^n\colon C^n \to C^{n+1}$ such that $d \circ d = 0$. $Z^n(C^\cdot) = \ker(d^n)$ is the module of *n-cocycles*, $B^n(C^\cdot) = \mathrm{im}(d^{n-1}) \subseteq C^n$ is the module of *n-coboundaries*, and the subquotient $H^n(C^\cdot) = Z^n/B^n$ of C^n is the n^{th} *cohomology module* of $C^\cdot$. Morphisms and quasi-isomorphisms of cochain complexes are defined exactly as for chain complexes.

A chain complex $C_.$ is called *bounded* if almost all the C_n are zero; if $C_n = 0$ unless $a \le n \le b$, we say that the complex has *amplitude* in $[a, b]$. A complex $C_.$ is *bounded above* (resp. *bounded below*) if there is a bound b (resp. a) such that $C_n = 0$ for all $n > b$ (resp. $n < a$). The bounded (resp. bounded above, resp. bounded below) chain complexes form full subcategories of **Ch** = **Ch**(R–**mod**) that are denoted $\mathbf{Ch}_b$, $\mathbf{Ch}_-$ and $\mathbf{Ch}_+$, respectively. The subcategory $\mathbf{Ch}_{\ge 0}$ of non-negative complexes $C_.$ ($C_n = 0$ for all $n < 0$) will be important in Chapter 8.

Similarly, a cochain complex $C^\cdot$ is called *bounded above* if the chain complex $C_.$ ($C_n = C^{-n}$) is bounded below, that is, if $C^n = 0$ for all large n; $C^\cdot$ is *bounded below* if $C_.$ is bounded above, and *bounded* if $C_.$ is bounded. The categories of bounded (resp. bounded above, resp. bounded below, resp. non-negative) cochain complexes are denoted $\mathbf{Ch}^b$, $\mathbf{Ch}^-$, $\mathbf{Ch}^+$, and $\mathbf{Ch}^{\ge 0}$, respectively.

Exercise 1.1.6 (Homology of a graph) Let Γ be a finite graph with V vertices $(v_1, \cdots, v_V)$ and E edges $(e_1, \cdots, e_E)$. If we orient the edges, we can form the *incidence matrix* of the graph. This is a $V \times E$ matrix whose (ij) entry is $+1$

if the edge e_j starts at v_i, -1 if e_j ends at v_i, and 0 otherwise. Let C_0 be the free R−module on the vertices, C_1 the free R−module on the edges, $C_n = 0$ if $n \neq 0, 1$, and $d: C_1 \to C_0$ be the incidence matrix. If Γ is connected (i.e., we can get from v_0 to every other vertex by tracing a path with edges), show that $H_0(C)$ and $H_1(C)$ are free R−modules of dimensions 1 and $V - E - 1$ respectively. (The number $V - E - 1$ is the number of *circuits* of the graph.) *Hint:* Choose basis $\{v_0, v_1 - v_0, \cdots, v_V - v_0\}$ for C_0, and use a path from v_0 to v_i to find an element of C_1 mapping to $v_i - v_0$.

Application 1.1.3 (Simplicial homology) Here is a topological application we shall discuss more in Chapter 8. Let K be a geometric simplicial complex, such as a triangulated polyhedron, and let K_k $(0 \leq k \leq n)$ denote the set of k-dimensional simplices of K. Each k-simplex has $k+1$ faces, which are ordered if the set K_0 of vertices is ordered (do so!), so we obtain $k+1$ set maps $\partial_i: K_k \to K_{k-1} (0 \leq i \leq k)$. The *simplicial chain complex* of K with coefficients in R is the chain complex $C_.$, formed as follows. Let C_k be the free R-module on the set K_k; set $C_k = 0$ unless $0 \leq k \leq n$. The set maps ∂_i yield $k+1$ module maps $C_k \to C_{k-1}$, which we also call ∂_i; their alternating sum $d = \sum(-1)^i \partial_i$ is the map $C_k \to C_{k-1}$ in the chain complex $C_.$. To see that $C_.$ is a chain complex, we need to prove the algebraic assertion that $d \circ d = 0$. This translates into the geometric fact that each $(k-2)$-dimensional simplex contained in a fixed k-simplex σ of K lies on exactly two faces of σ. The homology of the chain complex $C_.$ is called the *simplicial homology* of K with coefficients in R. This simplicial approach to homology was used in the first part of this century, before the advent of singular homology.

Exercise 1.1.7 (Tetrahedron) The tetrahedron T is a surface with 4 vertices, 6 edges, and 4 2-dimensional faces. Thus its homology is the homology of a chain complex $0 \to R^4 \to R^6 \to R^4 \to 0$. Write down the matrices in this complex and verify computationally that $H_2(T) \cong H_0(T) \cong R$ and $H_1(T) = 0$.

Application 1.1.4 (Singular homology) Let X be a topological space, and let $S_k = S_k(X)$ be the free R-module on the set of continuous maps from the standard k-simplex Δ_k to X. Restriction to the i^{th} face of Δ_k $(0 \leq i \leq k)$ transforms a map $\Delta_k \to X$ into a map $\Delta_{k-1} \to X$, and induces an R-module homomorphism ∂_i from S_k to S_{k-1}. The alternating sums $d = \sum(-1)^i \partial_i$ (from S_k to S_{k-1}) assemble to form a chain complex

$$\cdots \xrightarrow{d} S_2 \xrightarrow{d} S_1 \xrightarrow{d} S_0 \longrightarrow 0,$$