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Mixed Finite Element Methods and Applications

混合有限元方法和应用

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Preface

About 10 years ago, Mixed and Hybrid Finite Element Methods by F. Brezzi and M. Fortin went out of print and we were asked to allow a second printing. The world had evolved and we thought that a revision was due and that some topics had to be added to the book. For this task, D. Boffi joined the team and we began to write the improved version. It turned out that this meant doubling the number of pages and essentially producing a new book.

We hope that the result is now a better, self-contained, presentation of the underlying issues, either from linear algebra or from functional analysis. The presentation of the basic results should now be accessible to readers which are not familiar with functional analysis, although willing to invest some effort in understanding mathematical issues.

The scope of finite element approximations was extended to $H(\underline{\mathrm{curl}}; \Omega)$ and the three-dimensional cases are now fully covered. Tensor elements were also considered for elasticity problems. The approximation of eigenvalue problems has been included as well.

Moreover, new applications have been introduced: mixed elasticity and electromagnetism. New results have been added to already treated applications such as the Stokes problem or mixed formulations of elliptic problems. Even so, some topics have been merely addressed. This is, for example, the case of a posteriori estimators, Discontinuous Galerkin methods and new developments on virtual elements which would have required a long development in an already (too?) long book. Indeed, each of these topics could be the subject of a whole book. The analysis of mixed methods is also relevant to many applications such as mortar methods or contact problems which were also reduced to a few remarks. This does not mean that these are not important. We had to stop somewhere. Indeed, we took a long time to do so.

We thus hope that this book will provide a good starting point for all those interested in mixed (and related) finite element methods.

Pavia, Italy — D. Boffi and F. Brezzi
Québec, Canada — M. Fortin

Contents

Chapter 1
Variational Formulations and Finite Element Methods

Although we shall not define in this chapter mixed and hybrid (or other non-standard) finite element methods in a very precise way, we would like to situate them in a sufficiently clear setting. As we shall see, boundaries between different methods are sometimes rather fuzzy. This will not be a real drawback if we nevertheless know how to apply correctly the principles underlying their analysis.

After having briefly recalled some basic facts about classical methods, we shall present a few model problems. The study of these problems will be the kernel of this book. Thereafter, we rapidly recall basic principles of *duality theory* as this will be our starting point to introduce mixed methods. *Domain decomposition* methods (allied to duality) will lead us to hybrid methods. Then we shall briefly discuss modified variational formulations that can be used to obtain better stability properties for the discretised versions.

1.1 Classical Methods

We recall here, in a very simplified way, some facts about optimisation methods and the classical finite element method. Such an introduction cannot be complete and does not want to be. We refer the reader to [146] or [334], among others, where standard finite element methods are clearly exposed. We also refer to [167] where an exhaustive analysis of many of our model problems can be found.

Let us consider a very common situation where the solution of a physical problem minimises some functional (usually an "energy functional"), in a "well chosen" space of admissible functions V that we take for the moment as a Hilbert space:

$$\inf_{v \in V} J(v). \tag{1.1.1}$$

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If the functional $J(\cdot)$ is differentiable (cf. [184] for instance) the minimum (whenever it exists) will be characterised by *a variational equation*

$$\langle J'(u), v\rangle_{V'\times V} = 0, \quad \forall v \in V, \tag{1.1.2}$$

where $\langle \cdot,\cdot\rangle_{V'\times V}$ denotes duality between V and its topological dual V', the derivative $J'(u)$ at point u being considered as a linear form on V.

The classical Ritz's method to approximate the solution of (1.1.1) consists in choosing a finite dimensional subspace V_m of V, and then looking for $u_m \in V_m$ solution of the problem

$$\inf_{v_m \in V_m} J(v_m), \tag{1.1.3}$$

or, differentiating,

$$\langle J'(u_m), v_m\rangle_{V'\times V} = 0, \quad \forall v_m \in V_m. \tag{1.1.4}$$

Let us consider, to fix ideas, a quadratic functional

$$J(v) := \frac{1}{2}a(v,v) - L(v), \tag{1.1.5}$$

where $a(\cdot,\cdot)$ is a bilinear form on V, which we suppose continuous and symmetric, and $L(\cdot)$ a linear form on V. The variational equation (1.1.2) can then be written as

$$a(u,v) = L(v) \quad \forall v \in V, \tag{1.1.6}$$

while the discrete problem (1.1.4) becomes

$$a(u_m, v_m) = L(v_m), \quad \forall\, v_m \in V_m,\ u_m \in V_m. \tag{1.1.7}$$

If a basis $w_1, w_2, \ldots, w_m$ of V_m is chosen, and if we write

$$u_m = \sum_{i=1}^{m} \alpha_i w_i, \tag{1.1.8}$$

problem (1.1.7) is reduced to the solution of the linear system

$$\sum_{i=1}^{m} a_{ij}\alpha_i = b_j,\ 1 \le j \le m, \tag{1.1.9}$$

where we set

$$a_{ij} := a(w_i, w_j),\ b_j := L(w_j). \tag{1.1.10}$$

This formulation can be extended to the case where the bilinear form $a(\cdot,\cdot)$ *is not symmetric* and where problem (1.1.7) no longer corresponds to a minimisation problem. This is then usually called a Galerkin method. Let us recall that problems of type (1.1.7) will have a unique solution if, in particular, the bilinear form $a(\cdot,\cdot)$ is *coercive*, that is if there exists a positive real number α such that for all v in V

$$a(v,v) \geq \alpha \|v\|_V^2. \tag{1.1.11}$$

The above described methodology is very general and classical. We can consider the finite element method as a special case in the following sense.

The finite element method is a general technique to build finite dimensional subspaces of a Hilbert space V in order to apply the Ritz-Galerkin method to a variational problem.

This technique is based on a few simple ideas. The fundamental one is the *partition of the domain Ω* in which the problem is posed, into a set of "simple" sub-domains, called elements. These elements are usually triangles, quadrilaterals, tetrahedra, etc. A space V *of functions defined on Ω is then approximated* by "simple" functions, defined on each sub-domain with suitable matching conditions at interfaces. Simple functions are usually polynomials or functions obtained from polynomials by a change of variables.

This, of course, a very summarised way of defining finite elements and this is surely not the best way to understand it from the computational point of view. We shall come back to this in Chap. 2 with a much more workable approach.

The point that we want to emphasise here is the following. *A finite element method can only be considered in relation with a variational principle and a functional space. Changing the variational principle and the space in which it is posed leads to a different finite element approximation (even if the solution for the continuous problems can remain the same).*

In the remaining of this Chapter, we shall see how different variational formulations can be built for the same physical problem. Each of these formulations will lead to a new setting for finite element approximations. *The common point of the methods analysed in this book is that they are founded on a variational principle expressing an equilibrium* (saddle point) *condition* rather than on a minimisation principle. We shall now try to see, on some examples, how such equilibrium principles can be built.

1.2 Model Problems and Elementary Properties of Some Functional Spaces

The aim of this section is to introduce some notation and to present five model problems that will underlie almost all cases analysed in this book. They will be the Dirichlet problem for Laplace's equation, the linear elasticity problem, Stokes' problem, a fourth-order problem modelling the deflection of a thin clamped plate, and, finally, the time-harmonic Maxwell system. These problems are closely interrelated and methods to analyse them will also be.