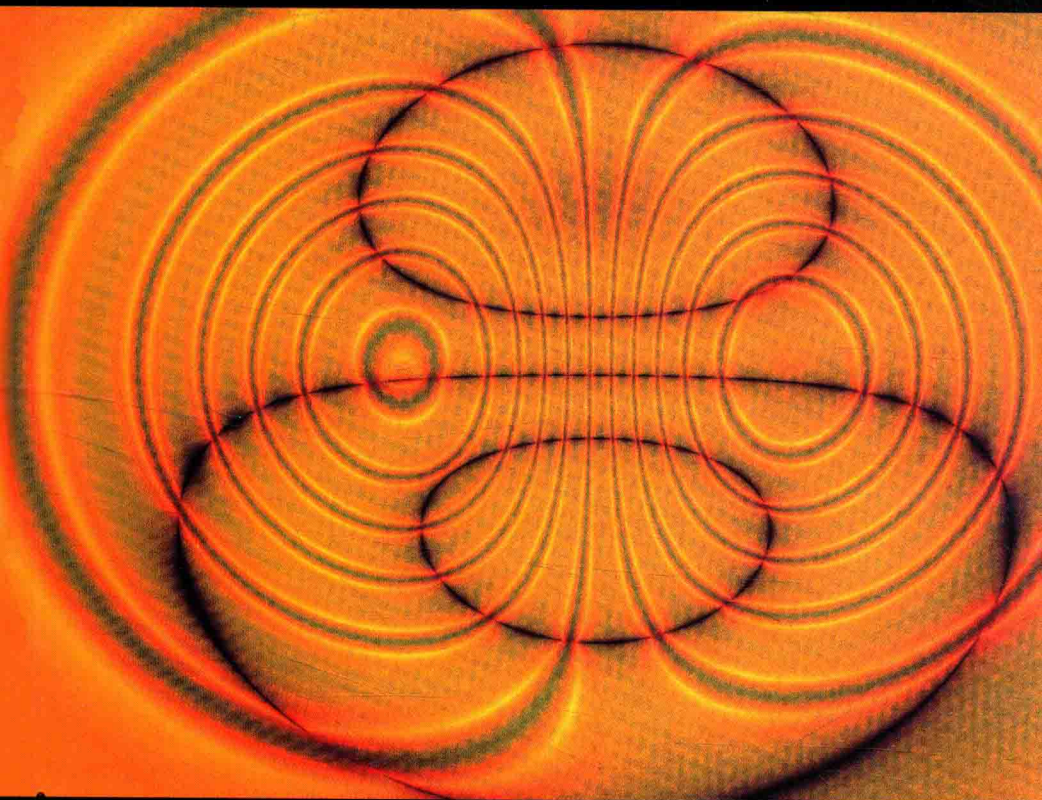


Morris W. Hirsch, Stephen Smale, Robert L. Devaney

# Differential Equations, Dynamical Systems, and an Introduction to Chaos

Third Edition

微分方程、动力系统与混沌引论 第3版



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# DIFFERENTIAL EQUATIONS, DYNAMICAL SYSTEMS, AND AN INTRODUCTION TO CHAOS

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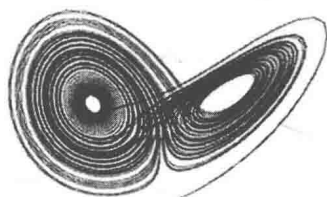
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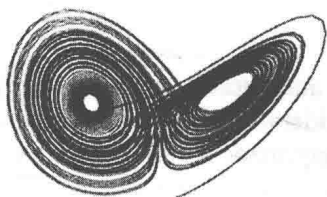




## Preface to Third Edition

The main new features in this edition consist of a number of additional explorations together with numerous proof simplifications and revisions. The new explorations include a sojourn into numerical methods that highlights how these methods sometimes fail, which in turn provides an early glimpse of chaotic behavior. Another new exploration involves the previously treated SIR model of infectious diseases, only now considered with zombies as the infected population. A third new exploration involves explaining the motion of a glider.

This edition has benefited from numerous helpful comments from a variety of readers. Special thanks are due to Jamil Gomes de Abreu, Eric Adams, Adam Leighton, Tiennyu Ma, Lluís Fernand Mello, Bogdan Przeradzki, Charles Pugh, Hal Smith, and Richard Venti for their valuable insights and corrections.



## Preface

In the thirty years since the publication of the first edition of this book, much has changed in the field of mathematics known as *dynamical systems*. In the early 1970s, we had very little access to high-speed computers and computer graphics. The word *chaos* had never been used in a mathematical setting. Most of the interest in the theory of differential equations and dynamical systems was confined to a relatively small group of mathematicians.

Things have changed dramatically in the ensuing three decades. Computers are everywhere, and software packages that can be used to approximate solutions of differential equations and view the results graphically are widely available. As a consequence, the analysis of nonlinear systems of differential equations is much more accessible than it once was. The discovery of complicated dynamical systems, such as the horseshoe map, homoclinic tangles, the Lorenz system, and their mathematical analysis, convinced scientists that simple stable motions such as equilibria or periodic solutions were not always the most important behavior of solutions of differential equations. The beauty and relative accessibility of these chaotic phenomena motivated scientists and engineers in many disciplines to look more carefully at the important differential equations in their own fields. In many cases, they found chaotic behavior in these systems as well.

Now dynamical systems phenomena appear in virtually every area of science, from the oscillating Belousov–Zhabotinsky reaction in chemistry to the chaotic Chua circuit in electrical engineering, from complicated motions in celestial mechanics to the bifurcations arising in ecological systems.

As a consequence, the audience for a text on differential equations and dynamical systems is considerably larger and more diverse than it was in the 1970s. We have accordingly made several major structural changes to this book, including:

1. The treatment of linear algebra has been scaled back. We have dispensed with the generalities involved with abstract vector spaces and normed linear spaces. We no longer include a complete proof of the reduction of all  $n \times n$  matrices to canonical form. Rather, we deal primarily with matrices no larger than  $4 \times 4$ .
2. We have included a detailed discussion of the chaotic behavior in the Lorenz attractor, the Shil'nikov system, and the double-scroll attractor.
3. Many new applications are included; previous applications have been updated.
4. There are now several chapters dealing with discrete dynamical systems.
5. We deal primarily with systems that are  $C^\infty$ , thereby simplifying many of the hypotheses of theorems.

This book consists of three main parts. The first deals with linear systems of differential equations together with some first-order nonlinear equations. The second is the main part of the text: here we concentrate on nonlinear systems, primarily two-dimensional, as well as applications of these systems in a wide variety of fields. Part three deals with higher dimensional systems. Here we emphasize the types of chaotic behavior that do not occur in planar systems, as well as the principal means of studying such behavior—the reduction to a discrete dynamical system.

Writing a book for a diverse audience whose backgrounds vary greatly poses a significant challenge. We view this one as a text for a second course in differential equations that is aimed not only at mathematicians, but also at scientists and engineers who are seeking to develop sufficient mathematical skills to analyze the types of differential equations that arise in their disciplines.

Many who come to this book will have strong backgrounds in linear algebra and real analysis, but others will have less exposure to these fields. To make this text accessible to both groups, we begin with a fairly gentle introduction to low-dimensional systems of differential equations. Much of this will be a review for readers with a more thorough background in differential equations, so we intersperse some new topics throughout the early part of the book for those readers.

For example, the first chapter deals with first-order equations. We begin it with a discussion of linear differential equations and the logistic population model, topics that should be familiar to anyone who has a rudimentary acquaintance with differential equations. Beyond this review, we discuss the logistic model with harvesting, both constant and periodic. This allows us to introduce bifurcations at an early stage as well as to describe Poincaré maps

and periodic solutions. These are topics that are not usually found in elementary differential equations courses, yet they are accessible to anyone with a background in multivariable calculus. Of course, readers with a limited background may wish to skip these specialized topics at first and concentrate on the more elementary material.

Chapters 2 through 6 deal with linear systems of differential equations. Again we begin slowly, with Chapters 2 and 3 dealing only with planar systems of differential equations and two-dimensional linear algebra. Chapters 5 and 6 introduce higher dimensional linear systems; however, our emphasis remains on three- and four-dimensional systems rather than completely general  $n$ -dimensional systems, even though many of the techniques we describe extend easily to higher dimensions.

The core of the book lies in the second part. Here, we turn our attention to nonlinear systems. Unlike linear systems, nonlinear systems present some serious theoretical difficulties such as existence and uniqueness of solutions, dependence of solutions on initial conditions and parameters, and the like. Rather than plunge immediately into these difficult theoretical questions, which require a solid background in real analysis, we simply state the important results in Chapter 7 and present a collection of examples that illustrate what these theorems say (and do not say). Proofs of all of the results are included in the final chapter of the book.

In the first few chapters in the nonlinear part of the book, we introduce important techniques such as linearization near equilibria, nullcline analysis, stability properties, limit sets, and bifurcation theory. In the latter half of this part, we apply these ideas to a variety of systems that arise in biology, electrical engineering, mechanics, and other fields.

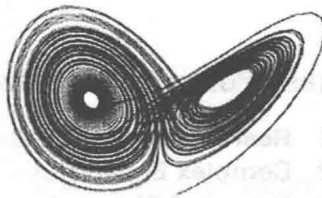
Many of the chapters conclude with a section called "Exploration." These sections consist of a series of questions and numerical investigations dealing with a particular topic or application relevant to the preceding material. In each Exploration we give a brief introduction to the topic at hand and provide references for further reading about this subject. But, we leave it to the reader to tackle the behavior of the resulting system using the material presented earlier. We often provide a series of introductory problems as well as hints as to how to proceed, but in many cases, a full analysis of the system could become a major research project. You will not find "answers in the back of the book" for the questions; in many cases, nobody knows the complete answer. (Except, of course, you!)

The final part of the book is devoted to the complicated nonlinear behavior of higher dimensional systems known as *chaotic behavior*. We introduce these ideas via the famous Lorenz system of differential equations. As is often the case in dimensions three and higher, we reduce the problem of comprehending the complicated behavior of this differential equation to that of understanding the dynamics of a discrete dynamical system or iterated



function. So we then take a detour into the world of discrete systems, discussing along the way how symbolic dynamics can be used to describe certain chaotic systems completely. We then return to nonlinear differential equations to apply these techniques to other chaotic systems, including those that arise when homoclinic orbits are present.

We maintain a website at [math.bu.edu/hsd](http://math.bu.edu/hsd) devoted to issues regarding this text. Look here for errata, suggestions, and other topics of interest to teachers and students of differential equations. We welcome any contributions from readers at this site.



# Contents

Preface to the Third Edition ix

Preface xi

## CHAPTER 1 First-Order Equations 1

- 1.1 The Simplest Example 1
- 1.2 The Logistic Population Model 4
- 1.3 Constant Harvesting and Bifurcations 7
- 1.4 Periodic Harvesting and Periodic Solutions 10
- 1.5 Computing the Poincaré Map 11
- 1.6 Exploration: A Two-Parameter Family 15

## CHAPTER 2 Planar Linear Systems 21

- 2.1 Second-Order Differential Equations 23
- 2.2 Planar Systems 24
- 2.3 Preliminaries from Algebra 26
- 2.4 Planar Linear Systems 29
- 2.5 Eigenvalues and Eigenvectors 30
- 2.6 Solving Linear Systems 33
- 2.7 The Linearity Principle 36

**CHAPTER 3 Phase Portraits for Planar Systems 39**

- 3.1 Real Distinct Eigenvalues 39
- 3.2 Complex Eigenvalues 44
- 3.3 Repeated Eigenvalues 47
- 3.4 Changing Coordinates 49

**CHAPTER 4 Classification of Planar Systems 61**

- 4.1 The Trace–Determinant Plane 61
- 4.2 Dynamical Classification 64
- 4.3 Exploration: A 3D Parameter Space 71

**CHAPTER 5 Higher-Dimensional Linear Algebra 73**

- 5.1 Preliminaries from Linear Algebra 73
- 5.2 Eigenvalues and Eigenvectors 82
- 5.3 Complex Eigenvalues 85
- 5.4 Bases and Subspaces 88
- 5.5 Repeated Eigenvalues 93
- 5.6 Genericity 100

**CHAPTER 6 Higher-Dimensional Linear Systems 107**

- 6.1 Distinct Eigenvalues 107
- 6.2 Harmonic Oscillators 114
- 6.3 Repeated Eigenvalues 120
- 6.4 The Exponential of a Matrix 123
- 6.5 Nonautonomous Linear Systems 130

**CHAPTER 7 Nonlinear Systems 139**

- 7.1 Dynamical Systems 140
- 7.2 The Existence and Uniqueness Theorem 142
- 7.3 Continuous Dependence of Solutions 147
- 7.4 The Variational Equation 149
- 7.5 Exploration: Numerical Methods 153
- 7.6 Exploration: Numerical Methods and Chaos 156

**CHAPTER 8 Equilibria in Nonlinear Systems 159**

- 8.1 Some Illustrative Examples 159
- 8.2 Nonlinear Sinks and Sources 165

- 8.3 Saddles 168
- 8.4 Stability 174
- 8.5 Bifurcations 175
- 8.6 Exploration: Complex Vector Fields 182

## CHAPTER 9 Global Nonlinear Techniques 187

- 9.1 Nullclines 187
- 9.2 Stability of Equilibria 192
- 9.3 Gradient Systems 202
- 9.4 Hamiltonian Systems 206
- 9.5 Exploration: The Pendulum with Constant Forcing 209

## CHAPTER 10 Closed Orbits and Limit Sets 213

- 10.1 Limit Sets 213
- 10.2 Local Sections and Flow Boxes 216
- 10.3 The Poincaré Map 218
- 10.4 Monotone Sequences in Planar Dynamical Systems 220
- 10.5 The Poincaré–Bendixson Theorem 222
- 10.6 Applications of Poincaré–Bendixson 225
- 10.7 Exploration: Chemical Reactions that Oscillate 228

## CHAPTER 11 Applications in Biology 233

- 11.1 Infectious Diseases 233
- 11.2 Predator–Prey Systems 237
- 11.3 Competitive Species 244
- 11.4 Exploration: Competition and Harvesting 250
- 11.5 Exploration: Adding Zombies to the SIR Model 251

## CHAPTER 12 Applications in Circuit Theory 257

- 12.1 An *RLC* Circuit 257
- 12.2 The Liénard Equation 261
- 12.3 The van der Pol Equation 263
- 12.4 A Hopf Bifurcation 270
- 12.5 Exploration: Neurodynamics 272

**CHAPTER 13 Applications in Mechanics 277**

- 13.1 Newton's Second Law 277
- 13.2 Conservative Systems 280
- 13.3 Central Force Fields 282
- 13.4 The Newtonian Central Force System 285
- 13.5 Kepler's First Law 290
- 13.6 The Two-Body Problem 293
- 13.7 Blowing Up the Singularity 294
- 13.8 Exploration: Other Central Force Problems 298
- 13.9 Exploration: Classical Limits of Quantum Mechanical Systems 299
- 13.10 Exploration: Motion of a Glider 301

**CHAPTER 14 The Lorenz System 305**

- 14.1 Introduction 306
- 14.2 Elementary Properties of the Lorenz System 308
- 14.3 The Lorenz Attractor 312
- 14.4 A Model for the Lorenz Attractor 316
- 14.5 The Chaotic Attractor 321
- 14.6 Exploration: The Rössler Attractor 326

**CHAPTER 15 Discrete Dynamical Systems 329**

- 15.1 Introduction 329
- 15.2 Bifurcations 334
- 15.3 The Discrete Logistic Model 337
- 15.4 Chaos 340
- 15.5 Symbolic Dynamics 344
- 15.6 The Shift Map 349
- 15.7 The Cantor Middle-Thirds Set 351
- 15.8 Exploration: Cubic Chaos 354
- 15.9 Exploration: The Orbit Diagram 355

**CHAPTER 16 Homoclinic Phenomena 361**

- 16.1 The Shilnikov System 361
- 16.2 The Horseshoe Map 368
- 16.3 The Double Scroll Attractor 375

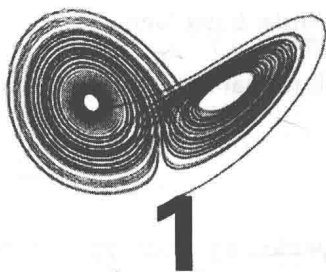
- 16.4 Homoclinic Bifurcations 377
- 16.5 Exploration: The Chua Circuit 381

## **CHAPTER 17 Existence and Uniqueness Revisited 385**

- 17.1 The Existence and Uniqueness Theorem 385
- 17.2 Proof of Existence and Uniqueness 387
- 17.3 Continuous Dependence on Initial Conditions 394
- 17.4 Extending Solutions 397
- 17.5 Nonautonomous Systems 401
- 17.6 Differentiability of the Flow 404

**Bibliography 411**

**Index 415**



# First-Order Equations

The purpose of this chapter is to develop some elementary yet important examples of first-order differential equations. The examples here illustrate some of the basic ideas in the theory of ordinary differential equations in the simplest possible setting.

We anticipate that the first few examples will be familiar to readers who have taken an introductory course in differential equations. Later examples, such as the logistic model with harvesting, are included to give the reader a taste of certain topics (e.g., bifurcations, periodic solutions, and Poincaré maps) that we will return to often throughout this book. In later chapters, our treatment of these topics will be much more systematic.

## 1.1 The Simplest Example

---

The differential equation familiar to all calculus students,

$$\frac{dx}{dt} = ax,$$

is the simplest. It is also one of the most important. First, what does it mean? Here  $x = x(t)$  is an unknown real-valued function of a real variable  $t$  and  $dx/dt$  is its derivative (we will also use  $x'$  or  $x'(t)$  for the derivative). In addition,  $a$  is a parameter; for each value of  $a$  we have a different differential

equation. The equation tells us that for every value of  $t$  the relationship

$$x'(t) = ax(t)$$

is true.

The solutions of this equation are obtained from calculus: if  $k$  is any real number, then the function  $x(t) = ke^{at}$  is a solution since

$$x'(t) = ake^{at} = ax(t).$$

Moreover, *there are no other solutions*. To see this, let  $u(t)$  be any solution and compute the derivative of  $u(t)e^{-at}$ :

$$\begin{aligned} \frac{d}{dt}(u(t)e^{-at}) &= u'(t)e^{-at} + u(t)(-ae^{-at}) \\ &= au(t)e^{-at} - au(t)e^{-at} = 0. \end{aligned}$$

Therefore,  $u(t)e^{-at}$  is a constant  $k$ , so  $u(t) = ke^{at}$ . This proves our assertion. Thus, we have found all possible solutions of this differential equation. We call the collection of all solutions of a differential equation the *general solution* of the equation.

The constant  $k$  appearing in this solution is completely determined if the value  $u_0$  of a solution at a single point  $t_0$  is specified. Suppose that a function  $x(t)$  satisfying the differential equation is also required to satisfy  $x(t_0) = u_0$ . Then we must have  $ke^{at_0} = u_0$ , so that  $k = u_0e^{-at_0}$ . Thus, we have determined  $k$  and this equation therefore has a unique solution satisfying the specified *initial condition*  $x(t_0) = u_0$ . For simplicity, we often take  $t_0 = 0$ ; then  $k = u_0$ . There is no loss of generality in taking  $t_0 = 0$ , for if  $u(t)$  is a solution with  $u(0) = u_0$ , then the function  $v(t) = u(t - t_0)$  is a solution with  $v(t_0) = u_0$ .

It is common to restate this in the form of an *initial value problem*:

$$x' = ax, \quad x(0) = u_0.$$

A solution  $x(t)$  of an initial value problem must not only solve the differential equation, but must also take on the prescribed initial value  $u_0$  at  $t = 0$ .

Note that there is a special solution of this differential equation when  $k = 0$ . This is the constant solution  $x(t) \equiv 0$ . A constant solution like this is called an *equilibrium solution* or *equilibrium point* for the equation. Equilibria are often among the most important solutions of differential equations.

The constant  $a$  in the equation  $x' = ax$  can be considered as a parameter. If  $a$  changes, the equation changes and so do the solutions. Can we describe qualitatively the way the solutions change? The sign of  $a$  is crucial here:

1. If  $a > 0$ ,  $\lim_{t \rightarrow \infty} ke^{at}$  equals  $\infty$  when  $k > 0$ , and equals  $-\infty$  when  $k < 0$



2. If  $a = 0$ ,  $ke^{at} = \text{constant}$
3. If  $a < 0$ ,  $\lim_{t \rightarrow \infty} ke^{at} = 0$

The qualitative behavior of solutions is vividly illustrated by sketching the graphs of solutions as in Figure 1.1.

Note that the behavior of solutions is quite different when  $a$  is positive and negative. When  $a > 0$ , all nonzero solutions tend away from the equilibrium point at 0 as  $t$  increases, whereas when  $a < 0$ , solutions tend toward the equilibrium point. We say that the equilibrium point is a *source* when nearby solutions tend away from it. The equilibrium point is a *sink* when nearby solutions tend toward it.

We also describe solutions by drawing them on the *phase line*. As the solution  $x(t)$  is a function of time, we may view  $x(t)$  as a particle moving along the real line. At the equilibrium point, the particle remains at rest (indicated by a solid dot), while any other solution moves up or down the  $x$ -axis, as indicated by the arrows in Figure 1.2.

The equation  $x' = ax$  is *stable* in a certain sense if  $a \neq 0$ . More precisely, if  $a$  is replaced by another constant  $b$  with a sign that is the same as  $a$ , then

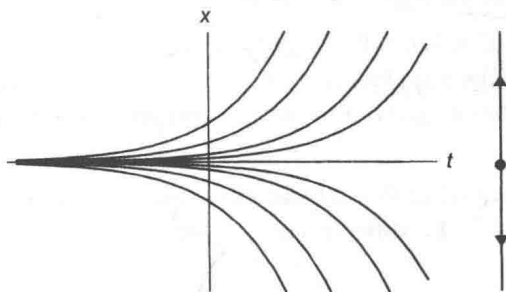


Figure 1.1 The solution graphs and phase line for  $x' = ax$  for  $a > 0$ . Each graph represents a particular solution.

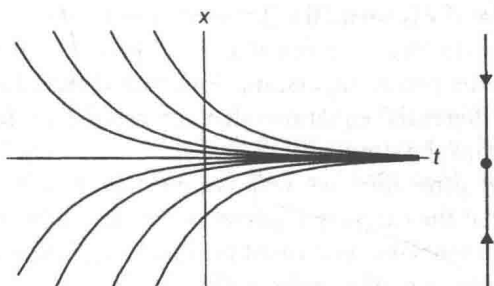


Figure 1.2 The solution graphs and phase line for  $x' = ax$  for  $a < 0$ .