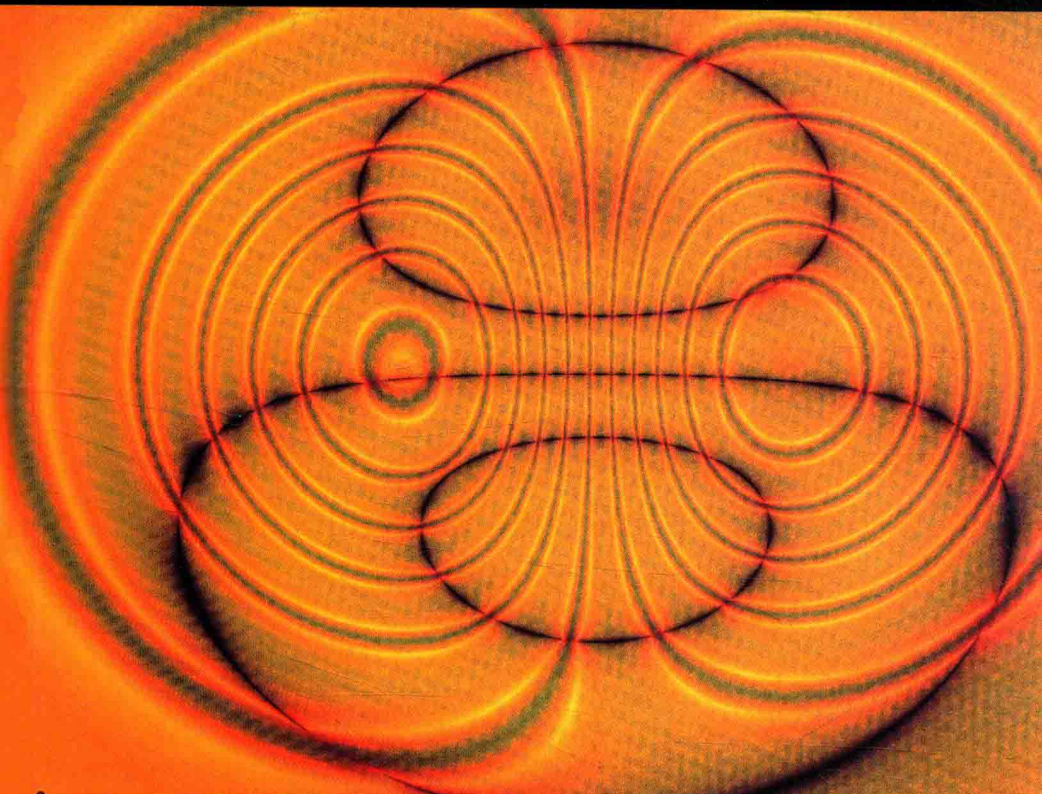
Morris W. Hirsch, Stephen Smale, Robert L. Devaney

Differential Equations, Dynamical Systems, and an Introduction to Chaos

Third Edition

微分方程、动力系统与混沌引论 第3版



Elsevier (Singapore) Pte Ltd.

世界图书出版公司
www.wpcbj.com.cn

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AMSTERDAM • BOSTON • HEIDELBERG • LONDON
NEW YORK • OXFORD • PARIS • SAN DIEGO
SAN FRANCISCO • SINGAPORE • SYDNEY • TOKYO
Academic Press is an imprint of Elsevier

图书在版编目（CIP）数据

微分方程、动力系统与混沌引论：第3版 = Differential Equations，Dynamical Systems，and an Introduction to Chaos Third Edition：英文/（美）赫希（Hirsch，M. W.）著．—影印本．—北京：世界图书出版公司北京公司，2015.8
ISBN 978-7-5192-0013-8

Ⅰ．①微… Ⅱ．①赫… Ⅲ．①微分方程—研究生—教材—英文 ②动力系统（数学）理论—研究生—教材—英文 ③混沌—研究生—教材—英文 Ⅳ．①O175 ②O19

中国版本图书馆 CIP 数据核字（2015）第207328号

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装帧设计：任志远

出版发行：世界图书出版公司北京公司
地　　址：北京市东城区朝内大街137号
邮　　编：100010
电　　话：010-64038355（发行）　64015580（客服）　64033507（总编室）
网　　址：http://www.wpcbj.com.cn
邮　　箱：wpcbjst@vip.163.com
销　　售：新华书店
印　　刷：三河市国英印务有限公司
开　　本：711mm×1245 mm　1/24
印　　张：18
字　　数：788千
版　　次：2016年1月第1版　　2016年1月第1次印刷
版权登记：01-2015-2758
ISBN 978-7-5192-0013-8　　　定价：75.00元

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ISBN: 978 - 0 - 12 - 382010 - 5

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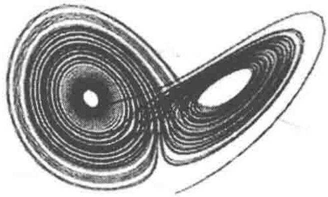
Tel: (65) 6349 - 0200
Fax: (65) 6733 - 1817

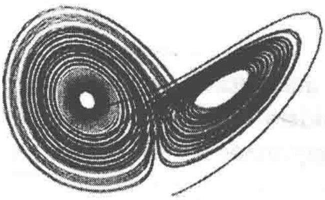
First Published 2015
2015 年初版



Preface to Third Edition

The main new features in this edition consist of a number of additional explorations together with numerous proof simplifications and revisions. The new explorations include a sojourn into numerical methods that highlights how these methods sometimes fail, which in turn provides an early glimpse of chaotic behavior. Another new exploration involves the previously treated SIR model of infectious diseases, only now considered with zombies as the infected population. A third new exploration involves explaining the motion of a glider.

This edition has benefited from numerous helpful comments from a variety of readers. Special thanks are due to Jamil Gomes de Abreu, Eric Adams, Adam Leighton, Tiennyu Ma, Lluis Fernand Mello, Bogdan Przeradzki, Charles Pugh, Hal Smith, and Richard Venti for their valuable insights and corrections.



Preface

In the thirty years since the publication of the first edition of this book, much has changed in the field of mathematics known as *dynamical systems*. In the early 1970s, we had very little access to high-speed computers and computer graphics. The word *chaos* had never been used in a mathematical setting. Most of the interest in the theory of differential equations and dynamical systems was confined to a relatively small group of mathematicians.

Things have changed dramatically in the ensuing three decades. Computers are everywhere, and software packages that can be used to approximate solutions of differential equations and view the results graphically are widely available. As a consequence, the analysis of nonlinear systems of differential equations is much more accessible than it once was. The discovery of complicated dynamical systems, such as the horseshoe map, homoclinic tangles, the Lorenz system, and their mathematical analysis, convinced scientists that simple stable motions such as equilibria or periodic solutions were not always the most important behavior of solutions of differential equations. The beauty and relative accessibility of these chaotic phenomena motivated scientists and engineers in many disciplines to look more carefully at the important differential equations in their own fields. In many cases, they found chaotic behavior in these systems as well.

Now dynamical systems phenomena appear in virtually every area of science, from the oscillating Belousov–Zhabotinsky reaction in chemistry to the chaotic Chua circuit in electrical engineering, from complicated motions in celestial mechanics to the bifurcations arising in ecological systems.

As a consequence, the audience for a text on differential equations and dynamical systems is considerably larger and more diverse than it was in the 1970s. We have accordingly made several major structural changes to this book, including:

1. The treatment of linear algebra has been scaled back. We have dispensed with the generalities involved with abstract vector spaces and normed linear spaces. We no longer include a complete proof of the reduction of all $n \times n$ matrices to canonical form. Rather, we deal primarily with matrices no larger than 4×4.
2. We have included a detailed discussion of the chaotic behavior in the Lorenz attractor, the Shil'nikov system, and the double-scroll attractor.
3. Many new applications are included; previous applications have been updated.
4. There are now several chapters dealing with discrete dynamical systems.
5. We deal primarily with systems that are C^∞, thereby simplifying many of the hypotheses of theorems.

This book consists of three main parts. The first deals with linear systems of differential equations together with some first-order nonlinear equations. The second is the main part of the text: here we concentrate on nonlinear systems, primarily two-dimensional, as well as applications of these systems in a wide variety of fields. Part three deals with higher dimensional systems. Here we emphasize the types of chaotic behavior that do not occur in planar systems, as well as the principal means of studying such behavior—the reduction to a discrete dynamical system.

Writing a book for a diverse audience whose backgrounds vary greatly poses a significant challenge. We view this one as a text for a second course in differential equations that is aimed not only at mathematicians, but also at scientists and engineers who are seeking to develop sufficient mathematical skills to analyze the types of differential equations that arise in their disciplines.

Many who come to this book will have strong backgrounds in linear algebra and real analysis, but others will have less exposure to these fields. To make this text accessible to both groups, we begin with a fairly gentle introduction to low-dimensional systems of differential equations. Much of this will be a review for readers with a more thorough background in differential equations, so we intersperse some new topics throughout the early part of the book for those readers.

For example, the first chapter deals with first-order equations. We begin it with a discussion of linear differential equations and the logistic population model, topics that should be familiar to anyone who has a rudimentary acquaintance with differential equations. Beyond this review, we discuss the logistic model with harvesting, both constant and periodic. This allows us to introduce bifurcations at an early stage as well as to describe Poincaré maps and periodic solutions. These are topics that are not usually found in elementary differential equations courses, yet they are accessible to anyone with a background in multivariable calculus. Of course, readers with a limited background may wish to skip these specialized topics at first and concentrate on the more elementary material.

Chapters 2 through 6 deal with linear systems of differential equations. Again we begin slowly, with Chapters 2 and 3 dealing only with planar systems of differential equations and two-dimensional linear algebra. Chapters 5 and 6 introduce higher dimensional linear systems; however, our emphasis remains on three- and four-dimensional systems rather than completely general n-dimensional systems, even though many of the techniques we describe extend easily to higher dimensions.

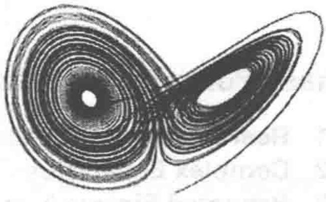
The core of the book lies in the second part. Here, we turn our attention to nonlinear systems. Unlike linear systems, nonlinear systems present some serious theoretical difficulties such as existence and uniqueness of solutions, dependence of solutions on initial conditions and parameters, and the like. Rather than plunge immediately into these difficult theoretical questions, which require a solid background in real analysis, we simply state the important results in Chapter 7 and present a collection of examples that illustrate what these theorems say (and do not say). Proofs of all of the results are included in the final chapter of the book.

In the first few chapters in the nonlinear part of the book, we introduce important techniques such as linearization near equilibria, nullcline analysis, stability properties, limit sets, and bifurcation theory. In the latter half of this part, we apply these ideas to a variety of systems that arise in biology, electrical engineering, mechanics, and other fields.

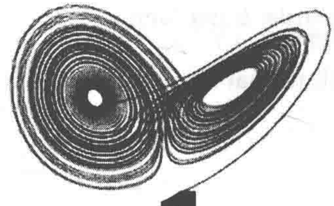
Many of the chapters conclude with a section called "Exploration." These sections consist of a series of questions and numerical investigations dealing with a particular topic or application relevant to the preceding material. In each Exploration we give a brief introduction to the topic at hand and provide references for further reading about this subject. But, we leave it to the reader to tackle the behavior of the resulting system using the material presented earlier. We often provide a series of introductory problems as well as hints as to how to proceed, but in many cases, a full analysis of the system could become a major research project. You will not find "answers in the back of the book" for the questions; in many cases, nobody knows the complete answer. (Except, of course, you!)

The final part of the book is devoted to the complicated nonlinear behavior of higher dimensional systems known as *chaotic behavior*. We introduce these ideas via the famous Lorenz system of differential equations. As is often the case in dimensions three and higher, we reduce the problem of comprehending the complicated behavior of this differential equation to that of understanding the dynamics of a discrete dynamical system or iterated function. So we then take a detour into the world of discrete systems, discussing along the way how symbolic dynamics can be used to describe certain chaotic systems completely. We then return to nonlinear differential equations to apply these techniques to other chaotic systems, including those that arise when homoclinic orbits are present.

We maintain a website at `math.bu.edu/hsd` devoted to issues regarding this text. Look here for errata, suggestions, and other topics of interest to teachers and students of differential equations. We welcome any contributions from readers at this site.



Contents



1 First-Order Equations

The purpose of this chapter is to develop some elementary yet important examples of first-order differential equations. The examples here illustrate some of the basic ideas in the theory of ordinary differential equations in the simplest possible setting.

We anticipate that the first few examples will be familiar to readers who have taken an introductory course in differential equations. Later examples, such as the logistic model with harvesting, are included to give the reader a taste of certain topics (e.g., bifurcations, periodic solutions, and Poincaré maps) that we will return to often throughout this book. In later chapters, our treatment of these topics will be much more systematic.

1.1 The Simplest Example

The differential equation familiar to all calculus students,

$$\frac{dx}{dt} = ax,$$

is the simplest. It is also one of the most important. First, what does it mean? Here $x = x(t)$ is an unknown real-valued function of a real variable t and dx/dt is its derivative (we will also use x' or $x'(t)$ for the derivative). In addition, a is a parameter; for each value of a we have a different differential equation. The equation tells us that for every value of t the relationship

$$x'(t) = ax(t)$$

is true.

The solutions of this equation are obtained from calculus: if k is any real number, then the function $x(t) = ke^{at}$ is a solution since

$$x'(t) = ake^{at} = ax(t).$$

Moreover, *there are no other solutions*. To see this, let $u(t)$ be any solution and compute the derivative of $u(t)e^{-at}$:

$$\begin{aligned}\frac{d}{dt}\left(u(t)e^{-at}\right) &= u'(t)e^{-at} + u(t)(-ae^{-at}) \\ &= au(t)e^{-at} - au(t)e^{-at} = 0.\end{aligned}$$

Therefore, $u(t)e^{-at}$ is a constant k, so $u(t) = ke^{at}$. This proves our assertion. Thus, we have found all possible solutions of this differential equation. We call the collection of all solutions of a differential equation the *general solution* of the equation.

The constant k appearing in this solution is completely determined if the value u_0 of a solution at a single point t_0 is specified. Suppose that a function $x(t)$ satisfying the differential equation is also required to satisfy $x(t_0) = u_0$. Then we must have $ke^{at_0} = u_0$, so that $k = u_0e^{-at_0}$. Thus, we have determined k and this equation therefore has a unique solution satisfying the specified *initial condition* $x(t_0) = u_0$. For simplicity, we often take $t_0 = 0$; then $k = u_0$. There is no loss of generality in taking $t_0 = 0$, for if $u(t)$ is a solution with $u(0) = u_0$, then the function $v(t) = u(t - t_0)$ is a solution with $v(t_0) = u_0$.

It is common to restate this in the form of an *initial value problem*:

$$x' = ax, \quad x(0) = u_0.$$

A solution $x(t)$ of an initial value problem must not only solve the differential equation, but must also take on the prescribed initial value u_0 at $t = 0$.

Note that there is a special solution of this differential equation when $k = 0$. This is the constant solution $x(t) \equiv 0$. A constant solution like this is called an *equilibrium solution* or *equilibrium point* for the equation. Equilibria are often among the most important solutions of differential equations.

The constant a in the equation $x' = ax$ can be considered as a parameter. If a changes, the equation changes and so do the solutions. Can we describe qualitatively the way the solutions change? The sign of a is crucial here:

1. If $a > 0$, $\lim_{t\to\infty} ke^{at}$ equals ∞ when $k > 0$, and equals $-\infty$ when $k < 0$

2. If $a = 0$, $ke^{at} = \text{constant}$
3. If $a < 0$, $\lim_{t\to\infty} ke^{at} = 0$

The qualitative behavior of solutions is vividly illustrated by sketching the graphs of solutions as in Figure 1.1.

Note that the behavior of solutions is quite different when a is positive and negative. When $a > 0$, all nonzero solutions tend away from the equilibrium point at 0 as t increases, whereas when $a < 0$, solutions tend toward the equilibrium point. We say that the equilibrium point is a *source* when nearby solutions tend away from it. The equilibrium point is a *sink* when nearby solutions tend toward it.

We also describe solutions by drawing them on the *phase line.* As the solution $x(t)$ is a function of time, we may view $x(t)$ as a particle moving along the real line. At the equilibrium point, the particle remains at rest (indicated by a solid dot), while any other solution moves up or down the x-axis, as indicated by the arrows in Figure 1.2.

The equation $x' = ax$ is *stable* in a certain sense if $a \neq 0$. More precisely, if a is replaced by another constant b with a sign that is the same as a, then

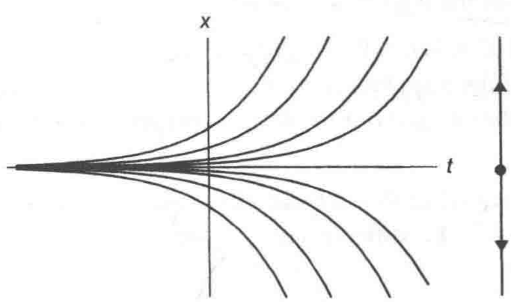


Figure 1.1 The solution graphs and phase line for $x' = ax$ for $a > 0$. Each graph represents a particular solution.

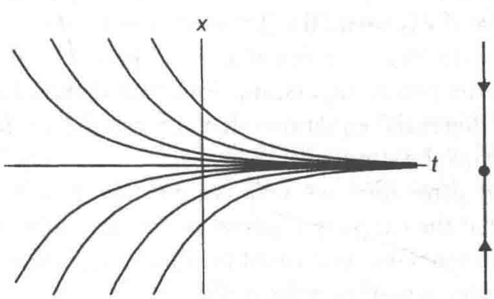


Figure 1.2 The solution graphs and phase line for $x' = ax$ for $a < 0$.