Undergraduate Texts in Mathematics

Kenneth A. Ross

Elementary Analysis
The Theory of Calculus
Second Edition

分析基础 微积分理论
第2版

Springer

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Undergraduate Texts in Mathematics are generally aimed at third- and fourth-year undergraduate mathematics students at North American universities. These texts strive to provide students and teachers with new perspectives and novel approaches. The books include motivation that guides the reader to an appreciation of interrelations among different aspects of the subject. They feature examples that illustrate key concepts as well as exercises that strengthen understanding.

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by Kenneth A. Ross

Preface

Preface to the First Edition A study of this book, and especially the exercises, should give the reader a thorough understanding of a few basic concepts in analysis such as continuity, convergence of sequences and series of numbers, and convergence of sequences and series of functions. An ability to read and write proofs will be stressed. A precise knowledge of definitions is essential. The beginner should memorize them; such memorization will help lead to understanding.

Chapter 1 sets the scene and, except for the completeness axiom, should be more or less familiar. Accordingly, readers and instructors are urged to move quickly through this chapter and refer back to it when necessary. The most critical sections in the book are §§7–12 in Chap. 2. If these sections are thoroughly digested and understood, the remainder of the book should be smooth sailing.

The first four chapters form a unit for a short course on analysis. I cover these four chapters (except for the enrichment sections and §20) in about 38 class periods; this includes time for quizzes and examinations. For such a short course, my philosophy is that the students are relatively comfortable with derivatives and integrals but do not really understand sequences and series, much less sequences and series of functions, so Chaps. 1–4 focus on these topics. On two or three occasions, I draw on the Fundamental Theorem of Calculus or the Mean Value Theorem, which appears later in the book, but of course these important theorems are at least discussed in a standard calculus class.

In the early sections, especially in Chap. 2, the proofs are very detailed with careful references for even the most elementary facts. Most sophisticated readers find excessive details and references a hindrance (they break the flow of the proof and tend to obscure the main ideas) and would prefer to check the items mentally as they proceed. Accordingly, in later chapters, the proofs will be somewhat less detailed, and references for the simplest facts will often be omitted. This should help prepare the reader for more advanced books which frequently give very brief arguments.

Mastery of the basic concepts in this book should make the analysis in such areas as complex variables, differential equations, numerical analysis, and statistics more meaningful. The book can also serve as a foundation for an in-depth study of real analysis given in books such as [4, 33, 34, 53, 62, 65] listed in the bibliography.

Readers planning to teach calculus will also benefit from a careful study of analysis. Even after studying this book (or writing it), it will not be easy to handle questions such as "What is a number?" but at least this book should help give a clearer picture of the subtleties to which such questions lead.

The enrichment sections contain discussions of some topics that I think are important or interesting. Sometimes the topic is dealt with lightly, and suggestions for further reading are given. Though these sections are not particularly designed for classroom use, I hope that some readers will use them to broaden their horizons and see how this material fits in the general scheme of things.

I have benefitted from numerous helpful suggestions from my colleagues Robert Freeman, William Kantor, Richard Koch, and John Leahy and from Timothy Hall, Gimli Khazad, and Jorge López. I have also had helpful conversations with my wife Lynn concerning grammar and taste. Of course, remaining errors in grammar and mathematics are the responsibility of the author.

Several users have supplied me with corrections and suggestions that I've incorporated in subsequent printings. I thank them all,

including Robert Messer of Albion College, who caught a subtle error in the proof of Theorem 12.1.

Preface to the Second Edition After 32 years, it seemed time to revise this book. Since the first edition was so successful, I have retained the format and material from the first edition. The numbering of theorems, examples, and exercises in each section will be the same, and new material will be added to some of the sections. Every rule has an exception, and this rule is no exception. In §11, a theorem (Theorem 11.2) has been added, which allows the simplification of four almost-identical proofs in the section: Examples 3 and 4, Theorem 11.7 (formerly Corollary 11.4), and Theorem 11.8 (formerly Theorem 11.7).

Where appropriate, the presentation has been improved. See especially the proof of the Chain Rule 28.4, the shorter proof of Abel's Theorem 26.6, and the shorter treatment of decimal expansions in §16. Also, a few examples have been added, a few exercises have been modified or added, and a couple of exercises have been deleted.

Here are the main additions to this revision. The proof of the irrationality of e in §16 is now accompanied by an elegant proof that π is also irrational. Even though this is an "enrichment" section, it is especially recommended for those who teach or will teach precollege mathematics. The Baire Category Theorem and interesting consequences have been added to the enrichment §21. Section 31, on Taylor's Theorem, has been overhauled. It now includes a discussion of Newton's method for approximating zeros of functions, as well as its cousin, the secant method. Proofs are provided for theorems that guarantee when these approximation methods work. Section 35 on Riemann-Stieltjes integrals has been improved and expanded. A new section, §38, contains an example of a continuous nowhere-differentiable function and a theorem that shows "most" continuous functions are nowhere differentiable. Also, each of §§22, 32, and 33 has been modestly enhanced.

It is a pleasure to thank many people who have helped over the years since the first edition appeared in 1980. This includes David M. Bloom, Robert B. Burckel, Kai Lai Chung, Mark Dalthorp (grandson), M. K. Das (India), Richard Dowds, Ray Hoobler, Richard M. Koch, Lisa J. Madsen, Pablo V. Negrón Marrero (Puerto Rico), Rajiv Monsurate (India), Theodore W. Palmer, Jürg Rätz (Switzerland), Peter Renz, Karl Stromberg, and Jesús Sueiras (Puerto Rico).

Special thanks go to my collaborator, Jorge M. López, who provided a huge amount of help and support with the revision. Working with him was also a lot of fun. My plan to revise the book was supported from the beginning by my wife, Ruth Madsen Ross. Finally, I thank my editor at Springer, Kaitlin Leach, who was attentive to my needs whenever they arose.

Especially for the Student: Don't be dismayed if you run into material that doesn't make sense, for whatever reason. It happens to all of us. Just tentatively accept the result as true, set it aside as something to return to, and forge ahead. Also, don't forget to use the Index or Symbols Index if some terminology or notation is puzzling.

Contents

1 CHAPTER Introduction

The underlying space for all the analysis in this book is the set of real numbers. In this chapter we set down some basic properties of this set. These properties will serve as our axioms in the sense that it is possible to derive all the properties of the real numbers using only these axioms. However, we will avoid getting bogged down in this endeavor. Some readers may wish to refer to the appendix on set notation.

§1 The Set $\mathbb{N}$ of Natural Numbers

We denote the set $\{1, 2, 3, \ldots\}$ of all *positive integers* by $\mathbb{N}$. Each positive integer n has a successor, namely $n + 1$. Thus the successor of 2 is 3, and 37 is the successor of 36. You will probably agree that the following properties of $\mathbb{N}$ are obvious; at least the first four are.

N1. 1 belongs to $\mathbb{N}$.
N2. If n belongs to $\mathbb{N}$, then its successor $n + 1$ belongs to $\mathbb{N}$.
N3. 1 is not the successor of any element in $\mathbb{N}$.

K.A. Ross, *Elementary Analysis: The Theory of Calculus*, Undergraduate Texts in Mathematics, DOI 10.1007/978-1-4614-6271-2_1,

N4. If n and m in $\mathbb{N}$ have the same successor, then $n = m$.
N5. A subset of $\mathbb{N}$ which contains 1, and which contains $n + 1$ whenever it contains n, must equal $\mathbb{N}$.

Properties N1 through N5 are known as the *Peano Axioms* or *Peano Postulates*. It turns out most familiar properties of $\mathbb{N}$ can be proved based on these five axioms; see [8] or [39].

Let's focus our attention on axiom N5, the one axiom that may not be obvious. Here is what the axiom is saying. Consider a subset S of $\mathbb{N}$ as described in N5. Then 1 belongs to S. Since S contains $n + 1$ whenever it contains n, it follows that S contains $2 = 1 + 1$. Again, since S contains $n + 1$ whenever it contains n, it follows that S contains $3 = 2 + 1$. Once again, since S contains $n + 1$ whenever it contains n, it follows that S contains $4 = 3+1$. We could continue this monotonous line of reasoning to conclude S contains any number in $\mathbb{N}$. Thus it seems reasonable to conclude $S = \mathbb{N}$. It is this reasonable conclusion that is asserted by axiom N5.

Here is another way to view axiom N5. Assume axiom N5 is false. Then $\mathbb{N}$ contains a set S such that

(i) $1 \in S$,
(ii) If $n \in S$, then $n + 1 \in S$,

and yet $S \neq \mathbb{N}$. Consider the smallest member of the set $\{n \in \mathbb{N} : n \notin S\}$, call it n_0. Since (i) holds, it is clear $n_0 \neq 1$. So n_0 is a successor to some number in $\mathbb{N}$, namely $n_0 - 1$. We have $n_0 - 1 \in S$ since n_0 is the smallest member of $\{n \in \mathbb{N} : n \notin S\}$. By (ii), the successor of $n_0 - 1$, namely n_0, is also in S, which is a contradiction. This discussion may be plausible, but we emphasize that we have not proved axiom N5 using the successor notion and axioms N1 through N4, because we implicitly used two unproven facts. We assumed every nonempty subset of $\mathbb{N}$ contains a least element and we assumed that if $n_0 \neq 1$ then n_0 is the successor to some number in $\mathbb{N}$.

Axiom N5 is the basis of mathematical induction. Let $P_1, P_2, P_3, \ldots$ be a list of statements or propositions that may or may not be true. The principle of mathematical induction asserts all the statements $P_1, P_2, P_3, \ldots$ are true provided

(I_1) P_1 is true,
(I_2) P_{n+1} is true whenever P_n is true.

We will refer to (I_1), i.e., the fact that P_1 is true, as the basis for induction and we will refer to (I_2) as the induction step. For a sound proof based on mathematical induction, properties (I_1) and (I_2) must both be verified. In practice, (I_1) will be easy to check.

Example 1

Prove $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$ for positive integers n. □

Solution

Our nth proposition is

$$P_n\text{: "}1 + 2 + \cdots + n = \frac{1}{2}n(n+1).\text{"}$$

Thus P_1 asserts $1 = \frac{1}{2} \cdot 1(1+1)$, P_2 asserts $1 + 2 = \frac{1}{2} \cdot 2(2+1)$, P_{37} asserts $1 + 2 + \cdots + 37 = \frac{1}{2} \cdot 37(37+1) = 703$, etc. In particular, P_1 is a true assertion which serves as our basis for induction.

For the induction step, suppose P_n is true. That is, we suppose

$$1 + 2 + \cdots + n = \tfrac{1}{2}n(n+1)$$

is true. Since we wish to prove P_{n+1} from this, we add $n+1$ to both sides to obtain

$$\begin{aligned} 1 + 2 + \cdots + n + (n+1) &= \tfrac{1}{2}n(n+1) + (n+1) \\ &= \tfrac{1}{2}[n(n+1) + 2(n+1)] = \tfrac{1}{2}(n+1)(n+2) \\ &= \tfrac{1}{2}(n+1)((n+1)+1). \end{aligned}$$

Thus P_{n+1} holds if P_n holds. By the principle of mathematical induction, we conclude P_n is true for all n. □

We emphasize that prior to the last sentence of our solution we did *not* prove "P_{n+1} is true." We merely proved an implication: "if P_n is true, then P_{n+1} is true." In a sense we proved an infinite number of assertions, namely: P_1 is true; if P_1 is true then P_2 is true; if P_2 is true then P_3 is true; if P_3 is true then P_4 is true; etc. Then we applied mathematical induction to conclude P_1 is true, P_2 is true, P_3 is true, P_4 is true, etc. We also confess that formulas like the one just proved are easier to prove than to discover. It can be a tricky matter to guess such a result. Sometimes results such as this are discovered by trial and error.

Example 2
All numbers of the form $5^n - 4n - 1$ are divisible by 16. □

Solution
More precisely, we show $5^n - 4n - 1$ is divisible by 16 for each n in $\mathbb{N}$. Our nth proposition is

$$P_n\text{: ``}5^n - 4n - 1 \quad \text{is divisible by} \quad 16.\text{''}$$

The basis for induction P_1 is clearly true, since $5^1 - 4 \cdot 1 - 1 = 0$. Proposition P_2 is also true because $5^2 - 4 \cdot 2 - 1 = 16$, but note we didn't need to check this case before proceeding to the induction step. For the induction step, suppose P_n is true. To verify P_{n+1}, the trick is to write

$$5^{n+1} - 4(n+1) - 1 = 5(5^n - 4n - 1) + 16n.$$

Since $5^n - 4n - 1$ is a multiple of 16 by the induction hypothesis, it follows that $5^{n+1} - 4(n+1) - 1$ is also a multiple of 16. In fact, if $5^n - 4n - 1 = 16m$, then $5^{n+1} - 4(n+1) - 1 = 16 \cdot (5m + n)$. We have shown P_n implies P_{n+1}, so the induction step holds. An application of mathematical induction completes the proof. □

Example 3
Show $|\sin nx| \leq n|\sin x|$ for all positive integers n and all real numbers x. □

Solution
Our nth proposition is

$$P_n\text{: ``}|\sin nx| \leq n|\sin x| \quad \text{for all real numbers} \quad x.\text{''}$$

The basis for induction is again clear. Suppose P_n is true. We apply the addition formula for sine to obtain

$$|\sin(n+1)x| = |\sin(nx + x)| = |\sin nx \cos x + \cos nx \sin x|.$$

Now we apply the Triangle Inequality and properties of the absolute value [see Theorems 3.7 and 3.5] to obtain

$$|\sin(n+1)x| \leq |\sin nx| \cdot |\cos x| + |\cos nx| \cdot |\sin x|.$$

Since $|\cos y| \leq 1$ for all y we see that

$$|\sin(n+1)x| \leq |\sin nx| + |\sin x|.$$