Springer Undergraduate Mathematics Series

Andrew Pressley

Elementary Differential Geometry

Second Edition

微分几何基础

第2版

Springer

世界图书出版公司
www.wpcbj.com.cn

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Springer

图书在版编目（CIP）数据

微分几何基础：第2版 = Elementary Differential Geometry Second Edition：英文/（英）普雷斯利（Pressley，A.）著．—影印本．—北京：世界图书出版公司北京公司，2015.8
ISBN 978-7-5192-0018-3

Ⅰ.①微…　Ⅱ.①普…　Ⅲ.①微分几何—英文　Ⅳ.①O186.1

中国版本图书馆 CIP 数据核字（2015）第207392号

Elementary Differential Geometry Second Edition
微分几何基础　第2版

著　　者：Andrew Pressley
责任编辑：刘　慧　岳利青
装帧设计：任志远

出版发行：世界图书出版公司北京公司
地　　址：北京市东城区朝内大街137号
邮　　编：100010
电　　话：010-64038355（发行）　64015580（客服）　64033507（总编室）
网　　址：http://www.wpcbj.com.cn
邮　　箱：wpcbjst@vip.163.com
销　　售：新华书店
印　　刷：三河市国英印务有限公司
开　　本：711mm×1245 mm　1/24
印　　张：20.5
字　　数：394千
版　　次：2016年1月第1版　　2016年1月第1次印刷
版权登记：01-2015-1564
ISBN 978-7-5192-0018-3　　　**定价**：75.00元

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Springer Undergraduate Mathematics Series ISSN 1615-2085
ISBN 978-1-84882-890-2 e-ISBN 978-1-84882-891-9
DOI 10.1007/978-1-84882-891-9
Springer London Dordrecht Heidelberg New York

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

Library of Congress Control Number: 2009942256

Mathematics Subject Classification (2000): 53-01, 53A04, 53A05, 53A35

Reprint from English language edition:
Elementary Differential Geometry Second Edition
by Andrew Pressley

Preface

The Differential Geometry in the title of this book is the study of the geometry of curves and surfaces in three-dimensional space using calculus techniques. This topic contains some of the most beautiful results in Mathematics, and yet most of them can be understood without extensive background knowledge. Thus, for virtually all of this book, the only pre-requisites are a good working knowledge of Calculus (including partial differentiation), Vectors and Linear Algebra (including matrices and determinants).

Many of the results about curves and surfaces that we shall discuss are prototypes of more general results that apply in higher-dimensional situations. For example, the Gauss–Bonnet theorem, treated in Chapter 11, is the prototype of a large number of results that relate 'local' and 'global' properties of geometric objects. The study of such relationships formed one of the major themes of 20th century Mathematics.

We want to emphasise, however, that the *methods* used in this book are *not* necessarily those which generalise to higher-dimensional situations. (For readers in the know, there is, for example, no mention of 'connections' in the remainder of this book.) Rather, we have tried at all times to use the simplest approach that will yield the desired results. Not only does this keep the pre-requisites to an absolute minimum, it also enables us to avoid some of the conceptual difficulties often encountered in the study of Differential Geometry in higher dimensions. We hope that this approach will make this beautiful subject accessible to a wider audience.

It is a cliché, but true nevertheless, that Mathematics can be learned only by doing it, and not just by reading about it. Accordingly, the book contains over 200 exercises. Readers should attempt as many of these as their stamina permits. Full solutions to all the exercises are given at the end of the book, but these should be consulted only after the reader has obtained his or her own solution, or in case of desperation. We have tried to minimise the number of instances of the latter by including hints to many of the less routine exercises.

Preface to the Second Edition

Few books get smaller when their second edition appears, and this is not one of those few. The largest addition is a new chapter devoted to hyperbolic (or non-Euclidean) geometry. Quite reasonably, most elementary treatments of this subject mimic Euclid's axiomatic treatment of ordinary plane geometry. A much quicker route to the main results is available, however, once the basics of the differential geometry of surfaces have been established, and it seemed a pity not to take advantage of it.

The other two most significant changes were suggested by commentators on the first edition. One was to treat the tangent plane more geometrically - this then allows one to define things like the first and second fundamental forms and the Weingarten map as geometric objects (rather than just as matrices). The second was to make use of parallel transport. I only partly agreed with this suggestion as I wanted to preserve the elementary nature of the book, but in this edition I have given a definition of parallel transport and related it to geodesics and Gaussian curvature. (However, for the experts reading this, I have stopped just short of introducing connections.)

There are many other smaller changes that are too numerous to list, but perhaps I should mention new sections on map-colouring (as an application of Gauss-Bonnet), and a self-contained treatment of spherical geometry. Apart from its intrinsic interest, spherical geometry provides the simplest 'non-Euclidean' geometry and it is in many respects analogous to its hyperbolic cousin. I have also corrected a number of errors in the first edition that were spotted either by me or by correspondents (mostly the latter).

For teachers thinking about using this book, I would suggest that there are now three routes through it that can be travelled in a single semester, terminating with *one* of chapters 11, 12 or 13, and taking in along the way the necessary basic material from chapters 1–10. For example, the new section on spherical geometry might be covered only if the final destination is hyperbolic geometry.

As in the first edition, solutions to all the exercises are provided at the end of the book. This feature was almost universally approved of by student commentators, and almost as universally disapproved of by teachers! Being one myself, I do understand the teachers' point of view, and to address it

I have devised a large number of new exercises that will be accessible online to all users of the book, together with a solutions manual for teachers, at www.springer.com.

I would like to thank all those who sent comments on the first edition, from beginning students through to experts - you know who you are! Even if I did not act on all your suggestions, I took them all seriously, and I hope that readers of this second edition will agree with me that the changes that resulted make the book more useful and more enjoyable (and not just longer).

Contents

Preface

Contents

1. Curves in the plane and in space

1.1 What is a curve? ... 1
1.2 Arc-length ... 9
1.3 Reparametrization ... 13
1.4 Closed curves ... 19
1.5 Level curves versus parametrized curves ... 23

2. How much does a curve curve?

2.1 Curvature ... 29
2.2 Plane curves ... 34
2.3 Space curves ... 46

3. Global properties of curves

3.1 Simple closed curves ... 55
3.2 The isoperimetric inequality ... 58
3.3 The four vertex theorem ... 62

4. Surfaces in three dimensions

4.1 What is a surface? ... 67
4.2 Smooth surfaces ... 76
4.3 Smooth maps ... 82
4.4 Tangents and derivatives ... 85
4.5 Normals and orientability ... 89

5. Examples of surfaces
5.1 Level surfaces 95
5.2 Quadric surfaces 97
5.3 Ruled surfaces and surfaces of revolution 104
5.4 Compact surfaces 109
5.5 Triply orthogonal systems 111
5.6 Applications of the inverse function theorem 116

6. The first fundamental form
6.1 Lengths of curves on surfaces 121
6.2 Isometries of surfaces 126
6.3 Conformal mappings of surfaces 133
6.4 Equiareal maps and a theorem of Archimedes 139
6.5 Spherical geometry 148

7. Curvature of surfaces
7.1 The second fundamental form 159
7.2 The Gauss and Weingarten maps 162
7.3 Normal and geodesic curvatures 165
7.4 Parallel transport and covariant derivative 170

8. Gaussian, mean and principal curvatures
8.1 Gaussian and mean curvatures 179
8.2 Principal curvatures of a surface 187
8.3 Surfaces of constant Gaussian curvature 196
8.4 Flat surfaces 201
8.5 Surfaces of constant mean curvature 206
8.6 Gaussian curvature of compact surfaces 212

9. Geodesics
9.1 Definition and basic properties 215
9.2 Geodesic equations 220
9.3 Geodesics on surfaces of revolution 227
9.4 Geodesics as shortest paths 235
9.5 Geodesic coordinates 242

10. Gauss' Theorema Egregium
10.1 The Gauss and Codazzi–Mainardi equations 247
10.2 Gauss' remarkable theorem 252
10.3 Surfaces of constant Gaussian curvature 257
10.4 Geodesic mappings 263

11. Hyperbolic geometry

11.1 Upper half-plane model 270
11.2 Isometries of $\mathcal{H}$ 277
11.3 Poincaré disc model 283
11.4 Hyperbolic parallels 290
11.5 Beltrami–Klein model 295

12. Minimal surfaces

12.1 Plateau's problem 305
12.2 Examples of minimal surfaces 312
12.3 Gauss map of a minimal surface 320
12.4 Conformal parametrization of minimal surfaces 322
12.5 Minimal surfaces and holomorphic functions 325

13. The Gauss–Bonnet theorem

13.1 Gauss–Bonnet for simple closed curves 335
13.2 Gauss–Bonnet for curvilinear polygons 342
13.3 Integration on compact surfaces 346
13.4 Gauss–Bonnet for compact surfaces 349
13.5 Map colouring 357
13.6 Holonomy and Gaussian curvature 362
13.7 Singularities of vector fields 365
13.8 Critical points 372

A0. Inner product spaces and self-adjoint linear maps

A1. Isometries of Euclidean spaces

A2. Möbius transformations

Hints to selected exercises

Solutions

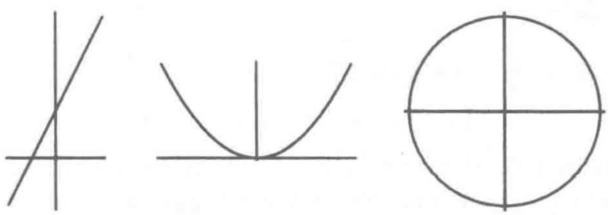
Index

1
Curves in the plane and in space

In this chapter, we discuss two mathematical formulations of the intuitive notion of a curve. The precise relation between them turns out to be quite subtle, so we begin by giving some examples of curves of each type and practical ways of passing between them.

1.1 What is a curve?

If asked to give an example of a curve, you might give a straight line, say $y - 2x = 1$ (even though this is not 'curved'!), or a circle, say $x^2 + y^2 = 1$, or perhaps a parabola, say $y - x^2 = 0$.



Andrew Pressley, *Elementary Differential Geometry: Second Edition*,
Springer Undergraduate Mathematics Series, DOI 10.1007/978-1-84882-891-9_1,

All of these curves are described by means of their Cartesian equation

$$f(x, y) = c,$$

where f is a function of x and y and c is a constant. From this point of view, a curve is a set of points, namely

$$\mathcal{C} = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = c\}. \quad (1.1)$$

These examples are all curves in the plane $\mathbb{R}^2$, but we can also consider curves in $\mathbb{R}^3$ – for example, the x-axis in $\mathbb{R}^3$ is the straight line given by

$$y = 0, \quad z = 0,$$

and more generally a curve in $\mathbb{R}^3$ might be defined by a pair of equations

$$f_1(x, y, z) = c_1, \qquad f_2(x, y, z) = c_2.$$

Curves of this kind are called *level curves*, the idea being that the curve in Eq. 1.1, for example, is the set of points (x, y) in the plane at which the quantity $f(x, y)$ reaches the 'level' c.

But there is another way to think about curves which turns out to be more useful in many situations. For this, a curve is viewed as the path traced out by a moving point. Thus, if $\gamma(t)$ is the position of the point at time t, the curve is described by a function γ of a scalar parameter t with vector values (in $\mathbb{R}^2$ for a plane curve, in $\mathbb{R}^3$ for a curve in space). We use this idea to give our first formal definition of a curve in $\mathbb{R}^n$ (we shall be interested only in the cases $n = 2$ or 3, but it is convenient to treat both cases simultaneously).

Definition 1.1.1

A *parametrized curve* in $\mathbb{R}^n$ is a map $\gamma : (\alpha, \beta) \to \mathbb{R}^n$, for some α, β with $-\infty \le \alpha < \beta \le \infty$.

The symbol (α, β) denotes the open interval

$$(\alpha, \beta) = \{t \in \mathbb{R} \mid \alpha < t < \beta\}.$$

A parametrized curve, whose image is contained in a level curve $\mathcal{C}$, is called a *parametrization* of (part of) $\mathcal{C}$. The following examples illustrate how to pass from level curves to parametrized curves and back again in practice.

Example 1.1.2

Let us find a parametrization $\gamma(t)$ of the parabola $y = x^2$. If $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, the components γ_1 and γ_2 of γ must satisfy

$$\gamma_2(t) = \gamma_1(t)^2 \quad (1.2)$$

for all values of t in the interval (α, β) where γ is defined (yet to be decided), and ideally every point on the parabola should be equal to $(\gamma_1(t), \gamma_2(t))$ for some value of $t \in (\alpha, \beta)$. Of course, there is an obvious solution to Eq. 1.2: take $\gamma_1(t) = t, \gamma_2(t) = t^2$. To get every point on the parabola we must allow t to take every real number value (since the x-coordinate of $\gamma(t)$ is just t, and the x-coordinate of a point on the parabola can be any real number), so we must take (α, β) to be $(-\infty, \infty)$. Thus, the desired parametrization is

$$\gamma : (-\infty, \infty) \to \mathbb{R}^2, \quad \gamma(t) = (t, t^2).$$

But this is not the only parametrization of the parabola. Another choice is $\gamma(t) = (t^3, t^6)$ (with $(\alpha, \beta) = (-\infty, \infty)$). Yet another is $(2t, 4t^2)$, and of course there are (infinitely many) others. So the parametrization of a given level curve is not unique.

Example 1.1.3

Now we try the circle $x^2 + y^2 = 1$. It is tempting to take $x = t$ as in the previous example, so that $y = \sqrt{1 - t^2}$ (we could have taken $y = -\sqrt{1 - t^2}$). So we get the parametrization

$$\gamma(t) = (t, \sqrt{1 - t^2}).$$

But this is only a parametrization of the upper half of the circle because $\sqrt{1 - t^2}$ is always ≥ 0. Similarly, if we had taken $y = -\sqrt{1 - t^2}$, we would only have covered the lower half of the circle.

If we want a parametrization of the whole circle, we must try again. We need functions $\gamma_1(t)$ and $\gamma_2(t)$ such that

$$\gamma_1(t)^2 + \gamma_2(t)^2 = 1 \tag{1.3}$$

for all $t \in (\alpha, \beta)$, and such that *every* point on the circle is equal to $(\gamma_1(t), \gamma_2(t))$ for some $t \in (\alpha, \beta)$. There is an obvious solution to Eq. 1.3: $\gamma_1(t) = \cos t$ and $\gamma_2(t) = \sin t$ (since $\cos^2 t + \sin^2 t = 1$ for all values of t). We can take $(\alpha, \beta) = (-\infty, \infty)$, although this is overkill: any open interval (α, β) whose length is greater than 2π will suffice.

The next example shows how to pass from parametrized curves to level curves.

Example 1.1.4

Take the parametrized curve (called an *astroid*)

$$\gamma(t) = (\cos^3 t, \sin^3 t), \quad t \in \mathbb{R}.$$

Since $\cos^2 t + \sin^2 t = 1$ for all t, the coordinates $x = \cos^3 t$, $y = \sin^3 t$ of the point $\gamma(t)$ satisfy

$$x^{2/3} + y^{2/3} = 1.$$

This level curve coincides with the image of the map γ. See Exercise 1.1.5 for a picture of the astroid.

In this book, we shall be studying parametrized curves (and later, surfaces) using methods of calculus. Such curves and surfaces will be described almost exclusively in terms of *smooth* functions: a function $f : (\alpha, \beta) \to \mathbb{R}$ is said to be smooth if the derivative $\frac{d^n f}{dt^n}$ exists for all $n \geq 1$ and all $t \in (\alpha, \beta)$. If $f(t)$ and $g(t)$ are smooth functions, it follows from standard results of calculus that the sum $f(t)+g(t)$, product $f(t)g(t)$, quotient $f(t)/g(t)$, and composite $f(g(t))$ are smooth functions, where they are defined.

To differentiate a *vector-valued* function such as $\gamma(t)$ (as in Definition 1.1.1), we differentiate componentwise: if

$$\gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t)),$$

then

$$\frac{d\gamma}{dt} = \left(\frac{d\gamma_1}{dt}, \frac{d\gamma_2}{dt}, \ldots, \frac{d\gamma_n}{dt}\right), \quad \frac{d^2\gamma}{dt^2} = \left(\frac{d^2\gamma_1}{dt^2}, \frac{d^2\gamma_2}{dt^2}, \ldots, \frac{d^2\gamma_n}{dt^2}\right), \quad \text{etc.}$$

To save space, we often denote $d\gamma/dt$ by $\dot{\gamma}(t)$, $d^2\gamma/dt^2$ by $\ddot{\gamma}(t)$, etc. We say that γ is *smooth* if the derivatives $d^n\gamma/dt^n$ exist for all $n \geq 1$ and all $t \in (\alpha, \beta)$; this is equivalent to requiring that each of the components $\gamma_1, \gamma_2, \ldots, \gamma_n$ of γ is smooth.

From now on, all parametrized curves studied in this book will be assumed to be smooth.

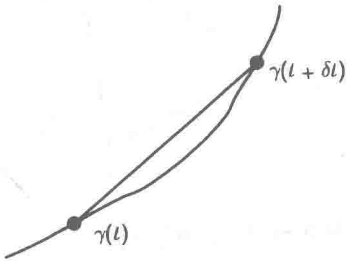
Definition 1.1.5

If γ is a parametrized curve, its first derivative $\dot{\gamma}(t)$ is called the *tangent vector* of γ at the point $\gamma(t)$.

To see the reason for this terminology, note that the vector

$$\frac{\gamma(t+\delta t)-\gamma(t)}{\delta t}$$

is parallel to the chord joining the points $\gamma(t)$ and $\gamma(t+\delta t)$ of the image $\mathcal{C}$ of γ:



As δt tends to zero the length of the chord also tends to zero, but we expect that the *direction* of the chord becomes parallel to that of the tangent to $\mathcal{C}$ at $\gamma(t)$. But the direction of the chord is the same as that of the vector

$$\frac{\gamma(t+\delta t)-\gamma(t)}{\delta t},$$

which tends to $d\gamma/dt$ as δt tends to zero. Of course, this only determines a well-defined direction tangent to the curve if $d\gamma/dt$ is non-zero. If that condition holds, we define the *tangent line* to $\mathcal{C}$ at a point $\mathbf{p}$ of $\mathcal{C}$ to be the straight line passing through $\mathbf{p}$ and parallel to the vector $d\gamma/dt$.

The following result is intuitively clear:

Proposition 1.1.6

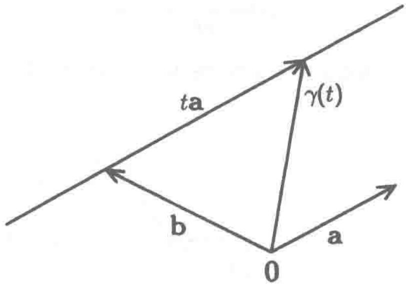
If the tangent vector of a parametrized curve is constant, the image of the curve is (part of) a straight line.

Proof

If $\dot{\gamma}(t) = \mathbf{a}$ for all t, where $\mathbf{a}$ is a constant vector, we have, integrating componentwise,

$$\gamma(t) = \int \frac{d\gamma}{dt}dt = \int \mathbf{a}\,dt = t\,\mathbf{a} + \mathbf{b},$$

where $\mathbf{b}$ is another constant vector. If $\mathbf{a} \neq 0$, this is the parametric equation of the straight line parallel to $\mathbf{a}$ and passing through the point $\mathbf{b}$:



If $\mathbf{a} = 0$, the image of γ is a single point (namely, $\mathbf{b}$). □

Before proceeding further with our study of curves, we should point out a potential source of confusion in the discussion of parametrized curves. This is regarding the question what is a 'point' of such a curve? The difficulty can be seen in the following example.

Example 1.1.7

The *limaçon* is the parametrized curve

$$\gamma(t) = ((1 + 2\cos t)\cos t, (1 + 2\cos t)\sin t), \qquad t \in \mathbb{R}$$

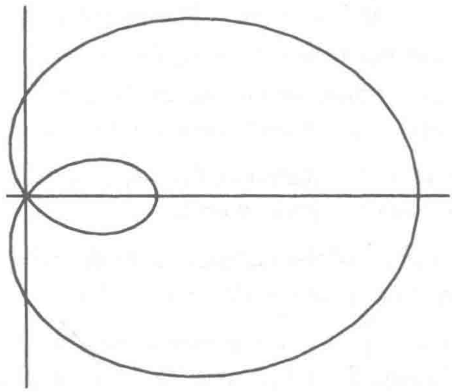
(see the diagram below). Note that γ has a self-intersection at the origin in the sense that $\gamma(t) = \mathbf{0}$ for $t = 2\pi/3$ and for $t = 4\pi/3$. The tangent vector is

$$\dot{\gamma}(t) = (-\sin t - 2\sin 2t, \cos t + 2\cos 2t).$$

In particular,

$$\dot{\gamma}(2\pi/3) = (\sqrt{3}/2, -3/2), \quad \dot{\gamma}(4\pi/3) = (-\sqrt{3}/2, -3/2).$$

So what is the tangent vector of this curve at the origin? Although $\dot{\gamma}(t)$ is well-defined for all values of t, it takes different values at $t = 2\pi/3$ and $t = 4\pi/3$, both of which correspond to the point $\mathbf{0}$ on the curve.



This example shows that we must be careful while talking about a 'point' of a parametrized curve γ: strictly speaking, this should be the same thing as a value of the curve parameter t, and not the corresponding geometric point $\gamma(t) \in \mathbb{R}^n$. Thus, Definition 1.1.5 should more properly read "If γ is a parametrized curve, its first derivative $\dot{\gamma}(t)$ is called the *tangent vector* of γ at the parameter value t." However, it seems to us that to insist on this distinction takes away from the geometric viewpoint, and we shall sometimes repeat the 'error' committed in the statement of Definition 1.1.5. This should not lead to confusion if the preceding remarks are kept in mind.

EXERCISES

1.1.1 Is $\gamma(t) = (t^2, t^4)$ a parametrization of the parabola $y = x^2$?

1.1.2 Find parametrizations of the following level curves:

(i) $y^2 - x^2 = 1$;

(ii) $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

1.1.3 Find the Cartesian equations of the following parametrized curves:

(i) $\gamma(t) = (\cos^2 t, \sin^2 t)$;

(ii) $\gamma(t) = (e^t, t^2)$.

1.1.4 Calculate the tangent vectors of the curves in Exercise 1.1.3.

1.1.5 Sketch the astroid in Example 1.1.4. Calculate its tangent vector at each point. At which points is the tangent vector zero?

1.1.6 Consider the ellipse

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1,$$