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J. Bergh J. Löfström

Interpolation Spaces

An Introduction

插值空间引论

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Preface

The works of Jaak Peetre constitute the main body of this treatise. Important contributors are also J.L. Lions and A.P. Calderón, not to mention several others. We, the present authors, have thus merely compiled and explained the works of others (with the exception of a few minor contributions of our own).

Let us mention the origin of this treatise. A couple of years ago, J. Peetre suggested to the second author, J. Löfström, writing a book on interpolation theory and he most generously put at Löfström's disposal an unfinished manuscript, covering parts of Chapter 1—3 and 5 of this book. Subsequently, Löfström prepared a first rough, but relatively complete manuscript of lecture notes. This was then partly rewritten and thouroughly revised by the first author, J. Bergh, who also prepared the notes and comment and most of the exercises.

Throughout the work, we have had the good fortune of enjoying Jaak Peetre's kind patronage and invaluable counsel. We want to express our deep gratitude to him. Thanks are also due to our colleagues for their support and help. Finally, we are sincerely grateful to Boel Engebrand, Lena Mattsson and Birgit Höglund for their expert typing of our manuscript.

This is the first attempt, as far as we know, to treat interpolation theory fairly comprehensively in book form. Perhaps this fact could partly excuse the many shortcomings, omissions and inconsistencies of which we may be guilty. We beg for all information about such insufficiencies and for any constructive criticism.

Lund and Göteborg, January 1976

Jöran Bergh Jörgen Löfström

Introduction

In recent years, there has emerged a new field of study in functional analysis: the theory of interpolation spaces. Interpolation theory has been applied to other branches of analysis (e.g. partial differential equations, numerical analysis, approximation theory), but it has also attracted considerable interest in itself. We intend to give an introduction to the theory, thereby covering the main elementary results.

In Chapter 1, we present the classical interpolation theorems of Riesz-Thorin and Marcinkiewicz with direct proofs, and also a few applications. The notation and the basic concepts are introduced in Chapter 2, where we also discuss some general results, e.g. the Aronszajn-Gagliardo theorem.

We treat two essentially different interpolation methods: the real method and the complex method. These two methods are modelled on the proofs of the Marcinkiewicz theorem and the Riesz-Thorin theorem respectively, as they are given in Chapter 1. The real method is presented, following Peetre, in Chapter 3; the complex method, following Calderón, in Chapter 4.

Chapter 5—7 contain applications of the general methods expounded in Chapter 3 and 4.

In Chapter 5, we consider interpolation of L_p-spaces, including general versions of the interpolation theorems of Riesz-Thorin, and of Marcinkiewicz, as well as other results, for instance, the theorem of Stein-Weiss concerning the interpolation of L_p-spaces with weights.

Chapter 6 contains the interpolation of Besov spaces and generalized Sobolev spaces (defined by means of Bessel potentials). We use the definition of the Besov spaces given by Peetre. We list the most important interpolation results for these spaces, and present various inclusion theorems, a general version of Sobolev's embedding theorem and a trace theorem. We also touch upon the theory of semi-groups of operators.

In Chapter 7 we discuss the close relation between interpolation theory and approximation theory (in a wide sense). We give some applications to classical approximation theory and theoretical numerical analysis.

We have emphasized the real method at the expense of a balance (with respect to applicability) between the real and the complex method. A reason for this is that the real interpolation theory, in contrast to the case of the complex theory, has not been treated comprehensively in one work. As a consequence, whenever it is possible to use both the real and the complex method, we have preferred to apply the real method.

In each chapter the penultimate section contains exercises. These are meant to extend and complement the results of the previous sections. Occasionally, we use the content of an exercise in the subsequent main text. We have tried to give references for the exercises. Moreover, many important results and most of the applications can be found only as exercises.

Concluding each chapter, we have a section with notes and comment. These include historical sketches, various generalizations, related questions and references. However, we have not aimed at completeness: the historical references are not necessarily the first ones; many papers worth mention have been left out. By giving a few key references, i.e. those which are pertinent to the reader's own further study, we hope to compensate partly for this.

The potential reader we have had in mind is conversant with the elements of real (several variables) and complex (one variable) analysis, of Fourier analysis, and of functional analysis. Beyond an elementary level, we have tried to supply proofs of the statements in the main text. Our general reference for elementary results is Dunford-Schwartz [1].

We use some symbols with a special convention or meaning. For other notation, see the Index of Symbols.

$f(x) \sim g(x)$ "There are positive constants C_1 and C_2 such that $C_1 g(x) \leqslant f(x) \leqslant C_2 g(x)$ (f and g being non-negative functions)."
Read: f and g are equivalent.

$T: A \to B$ "T is a continuous mapping from A to B."

$A \subset B$ "A is continuously embedded in B."

Table of Contents

Chapter 1

Some Classical Theorems

The classical results which provided the main impetus for the study of interpolation *in se* are the theorems of M. Riesz, with Thorin's proof, and of Marcinkiewicz. Thorin's proof of the Riesz-Thorin theorem contains the idea behind the complex interpolation method. Analogously, the way of proving the Marcinkiewicz theorem resembles the construction of the real interpolation method. We give direct proofs of these theorems (Section 1.1 and Section 1.3), and a few of their applications (Section 1.2 and Section 1.4). More recently, interpolation methods have been used in approximation theory. In Section 1.5 we rewrite the classical Bernstein and Jackson inequalities to indicate the connection with approximation theory.

The purpose of this chapter is to introduce the type of theorems which will be proved later, and also to give a first hint of the techniques used in their proofs. Note that, in this introductory chapter, we are not stating the results in the more general form they will have in later chapters.

1.1. The Riesz-Thorin Theorem

Let (U, μ) be a measure space, μ always being a positive measure. We adopt the usual convention that two functions are considered equal if they agree except on a set of μ-measure zero. Then we denote by $L_p(U, d\mu)$ (or simply $L_p(d\mu)$, $L_p(U)$ or even L_p) the Lebesgue-space of (all equivalence classes of) scalar-valued μ-measurable functions f on U, such that

(1) $$\|f\|_{L_p}=(\int_U |f(x)|^p d\mu)^{1/p}$$

is finite. Here we have $1 \leqslant p < \infty$. In the limiting case, $p=\infty$, L_p consists of all μ-measurable and bounded functions. Then we write

(2) $$\|f\|_{L_\infty}=\sup_U |f(x)| .$$

In this section and the next, scalars are supposed to be complex numbers.

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Let T be a linear mapping from $L_p = L_p(U, d\mu)$ to $L_q = L_q(V, d\nu)$. This means that $T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$. We shall write

$$T: L_p \to L_q$$

if in addition T is bounded, i.e. if

$$M = \sup_{f \neq 0} \|Tf\|_{L_q} / \|f\|_{L_p}$$

is finite. The number M is called the norm of the mapping T.

Now we have the following well-known theorem.

1.1.1. Theorem (The Riesz-Thorin interpolation theorem). *Assume that* $p_0 \neq p_1$, $q_0 \neq q_1$ *and that*

$$T: L_{p_0}(U, d\mu) \to L_{q_0}(V, d\nu)$$

with norm M_0, *and that*

$$T: L_{p_1}(U, d\mu) \to L_{q_1}(V, d\nu)$$

with norm M_1. *Then*

$$T: L_p(U, d\mu) \to L_q(V, d\nu)$$

with norm

(3) $$M \leqslant M_0^{1-\theta} M_1^{\theta}$$

provided that $0 < \theta < 1$ *and*

(4) $$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Note that (3) means that M is logarithmically convex, i.e. $\log M$ is convex. Note also the geometrical meaning of (4). The points $(1/p, 1/q)$ described by (4)

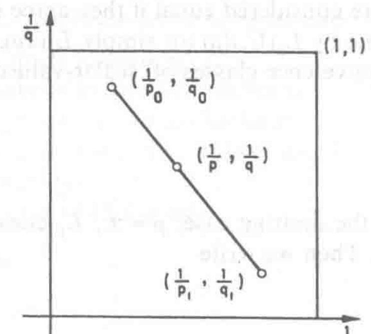

Fig. 1

are the points on the line segment between $(1/p_0, 1/q_0)$ and $(1/p_1, 1/q_1)$. (Obviously one should think of L_p as a "function" of $1/p$ rather than of p.)

Later on we shall prove the Riesz-Thorin interpolation (or convexity) theorem by means of abstract methods. Here we shall reproduce the elementary proof which was given by Thorin.

Proof: Let us write

$$\langle h, g\rangle = \int_V h(y)g(y)dv$$

and $1/q' = 1 - 1/q$. Then we have, by Hölder's inequality,

$$\|h\|_{L_q} = \sup\{|\langle h, g\rangle| : \|g\|_{L_{q'}} = 1\}.$$

and

$$M = \sup\{|\langle Tf, g\rangle| : \|f\|_{L_p} = \|g\|_{L_{q'}} = 1\}.$$

Since $p < \infty, q' < \infty$ we can assume that f and g are bounded with compact supports.

For $0 \leqslant \operatorname{Re} z \leqslant 1$ we put

$$\frac{1}{p(z)} = \frac{1-z}{p_0} + \frac{z}{p_1}, \qquad \frac{1}{q'(z)} = \frac{1-z}{q'_0} + \frac{z}{q'_1},$$

and

$$\varphi(z) = \varphi(x, z) = |f(x)|^{p/p(z)} f(x)/|f(x)|, \qquad x \in U,$$
$$\psi(z) = \psi(y, z) = |g(y)|^{q'/q'(z)} g(y)/|g(y)|, \qquad y \in V.$$

It follows that $\varphi(z) \in L_{p_j}$ and $\psi(z) \in L_{q'_j}$ and hence that $T\varphi(z) \in L_{q_j}$, $j = 0, 1$. It is also easy to see that $\varphi'(z) \in L_{p_j}$, $\psi'(z) \in L_{q_j}$ and thus also that $(T\varphi)'(z) \in L_{q_j}$, $(0 < \operatorname{Re} z < 1)$. This implies the existence of

$$F(z) = \langle T\varphi(z), \psi(z)\rangle, \qquad 0 \leqslant \operatorname{Re} z \leqslant 1.$$

Moreover it follows that $F(z)$ is analytic on the open strip $0 < \operatorname{Re} z < 1$, and bounded and continuous on the closed strip $0 \leqslant \operatorname{Re} z \leqslant 1$.

Next we note that

$$\|\varphi(it)\|_{L_{p_0}} = \|\,|f|^{p/p_0}\|_{L_{p_0}} = \|f\|_{L_p}^{p/p_0} = 1,$$
$$\|\varphi(1+it)\|_{L_{p_1}} = \|\,|f|^{p/p_1}\|_{L_{p_1}} = \|f\|_{L_p}^{p/p_1} = 1,$$

and similarly

$$\|\psi(it)\|_{L_{q'_0}} = \|\psi(1+it)\|_{L_{q'_1}} = 1.$$

By the assumptions, we therefore have

$$(3)\qquad \begin{aligned}|F(it)| &\leqslant \| T\varphi(it)\|_{L_{q_0}} \cdot \|\psi(it)\|_{L_{q_0'}} \leqslant M_0\,,\\ |F(1+it)| &\leqslant \| T\varphi(1+it)\|_{L_{q_1}} \cdot \|\psi(1+it)\|_{L_{q_1'}} \leqslant M_1\,.\end{aligned}$$

We also note that

$$\varphi(\theta)=f\,, \qquad \psi(\theta)=g\,,$$

and thus

$$(4)\qquad F(\theta)=\langle Tf, g\rangle\,.$$

Using now the three line theorem (a variant of the well-known Hadamard three circle theorem), reproduced as Lemma 1.1.2 below, we get the conclusion

$$|\langle Tf, g\rangle| \leqslant M_0^{1-\theta} M_1^{\theta}\,,$$

or equivalently

$$M \leqslant M_0^{1-\theta} M_1^{\theta}\,. \quad \square$$

1.1.2. Lemma (The three line theorem). *Assume that $F(z)$ is analytic on the open strip $0<\operatorname{Re} z<1$ and bounded and continuous on the closed strip $0 \leqslant \operatorname{Re} z \leqslant 1$. If*

$$|F(it)| \leqslant M_0\,, \qquad |F(1+it)| \leqslant M_1\,, \qquad -\infty<t<\infty\,,$$

we then have

$$|F(\theta+it)| \leqslant M_0^{1-\theta} M_1^{\theta}\,, \qquad -\infty<t<\infty\,.$$

Proof: Let ε be a positive and λ an arbitrary real number. Put

$$F_\varepsilon(z)=\exp(\varepsilon z^2+\lambda z)\, F(z)\,.$$

Then it follows that

$$F_\varepsilon(z) \to 0 \quad \text{as} \quad \operatorname{Im} z \to \pm\infty\,,$$

and

$$|F_\varepsilon(it)| \leqslant M_0\,, \qquad |F_\varepsilon(1+it)| \leqslant M_1 e^{\varepsilon+\lambda}\,.$$

By the Phragmén-Lindelöf principle we therefore obtain

$$|F_\varepsilon(z)| \leqslant \max(M_0, M_1 e^{\varepsilon+\lambda})\,,$$

i.e.,

$$|F(\theta+it)| \leqslant \exp(-\varepsilon(\theta^2-t^2)) \max(M_0 e^{-\theta\lambda}, M_1 e^{(1-\theta)\lambda+\varepsilon})\,.$$

This holds for any fixed θ and t. Letting $\varepsilon \to 0$ we conclude that

$$|F(\theta+it)| \leqslant \max(M_0\rho^{-\theta}, M_1\rho^{1-\theta})$$

where $\rho = \exp\lambda$. The right hand side is as small as possible when $M_0\rho^{-\theta} = M_1\rho^{1-\theta}$, i.e. when $\rho = M_0/M_1$. With this choice of ρ we get

$$|F(\theta+it)| \leqslant M_0^{1-\theta} M_1^{\theta}. \quad \square$$

1.2. Applications of the Riesz-Thorin Theorem

In this section we shall give two rather simple applications of the Riesz-Thorin interpolation theorem. We include them here in order to illustrate the rôle of interpolation theorems of which the Riesz-Thorin theorem is just one (albeit important) example.

We shall consider the case $U = V = \mathbb{R}^n$ and $d\mu = d\nu = dx$ (Lebesgue-measure). We let T be the Fourier transform $\mathscr{F}$ defined by

$$(\mathscr{F}f)(\xi) = \hat{f}(\xi) = \int f(x)\exp(-i\langle x, \xi\rangle)dx,$$

where $\langle x, \xi\rangle = x_1\xi_1 + \cdots + x_n\xi_n$. Here $x = (x_1, \ldots, x_n)$ and $\xi = (\xi_1, \ldots, \xi_n)$. Then we have

$$|\mathscr{F}f(\xi)| \leqslant \int |f(x)|dx$$

and by Parseval's formula

$$\int |\mathscr{F}f(\xi)|^2 d\xi = (2\pi)^n \int |f(x)|^2 dx.$$

This means that

$$\mathscr{F}: L_1 \to L_\infty, \quad \text{norm } 1,$$

$$\mathscr{F}: L_2 \to L_2, \quad \text{norm } (2\pi)^{n/2}.$$

Using the Riesz-Thorin theorem, we conclude that

(1) $$\mathscr{F}: L_p \to L_q$$

with

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{2}, \quad \frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{2}, \quad 0 < \theta < 1.$$

Eliminating θ, we see that $1/p = 1 - 1/q$, i.e., $q = p'$, where $1 < p < 2$. The norm of the mapping (1) is bounded by $(2\pi)^{n\theta/2} = (2\pi)^{n/p'}$. We have proved the following result.

1.2.1. Theorem (The Hausdorff-Young inequality). *If* $1 \leqslant p \leqslant 2$ *we have*

$$\|\mathscr{F}f\|_{L_{p'}} \leqslant (2\pi)^{n/p'} \|f\|_{L_p}. \quad \square$$

As a second application of the Riesz-Thorin theorem we consider the convolution operator

$$Tf(x) = \int k(x-y) f(y) dy = k * f(x)$$

where k is a given function in L_ρ. By Minkowski's inequality we have

$$\|Tf\|_{L_\rho} \leqslant \|k\|_{L_\rho} \|f\|_{L_1},$$

and, by Hölder's inequality,

$$\|Tf\|_{L_\infty} \leqslant \|k\|_{L_\rho} \|f\|_{L_{\rho'}}.$$

Thus

$$T\colon L_1 \to L_\rho,$$

$$T\colon L_{\rho'} \to L_\infty,$$

and therefore

$$T\colon L_p \to L_q$$

where

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{\rho'}, \qquad \frac{1}{q} = \frac{1-\theta}{\rho} + \frac{\theta}{\infty}.$$

Elimination of θ yields $1/q = 1/p - 1/\rho'$ and $1 \leqslant p \leqslant \rho'$. This gives the following result.

1.2.2. Theorem (Young's inequality). *If* $k \in L_\rho$ *and* $f \in L_p$ *where* $1 < p < \rho'$ *then* $k * f \in L_q$ *for* $1/q = 1/p - 1/\rho'$ *and*

$$\|k * f\|_{L_q} \leqslant \|k\|_{L_\rho} \|f\|_{L_p}. \quad \square$$

1.3. The Marcinkiewicz Theorem

Consider again the measure space (U, μ). In this section the scalars may be real or complex. If f is a scalar-valued μ-measurable function which is finite almost everywhere, we introduce the distribution function $m(\sigma, f)$ defined by

$$m(\sigma, f) = \mu(\{x\colon |f(x)| > \sigma\}).$$

Since we have assumed that μ is positive, we have that $m(\sigma, f)$ is a real-valued or extended real-valued function of σ, defined on the positive real axis $\mathbb{R}_+ = (0, \infty)$. Clearly $m(\sigma, f)$ is non-increasing and continuous on the right. Moreover, we have

$$(1) \qquad \|f\|_{L_p} = (p \int_0^\infty \sigma^p m(\sigma, f)\, d\sigma/\sigma)^{1/p} \quad \text{if} \quad 1 \leqslant p < \infty\,,$$

and

$$(2) \qquad \|f\|_{L_\infty} = \inf\{\sigma : m(\sigma, f) = 0\}\,.$$

Using the distribution function $m(\sigma, f)$, we now introduce the weak L_p-spaces denoted by L_p^*. The space L_p^*, $1 \leqslant p < \infty$, consists of all f such that

$$\|f\|_{L_p^*} = \sup_\sigma \sigma m(\sigma, f)^{1/p} < \infty\,.$$

In the limiting case $p = \infty$ we put $L_\infty^* = L_\infty$. Note that $\|f\|_{L_p^*}$ is not a norm if $1 \leqslant p < \infty$. In fact, it is clear that

$$(3) \qquad m(\sigma, f+g) \leqslant m(\sigma/2, f) + m(\sigma/2, g)\,.$$

Using the inequality $(a+b)^{1/p} \leqslant a^{1/p} + b^{1/p}$, we conclude that

$$\|f+g\|_{L_p^*} \leqslant 2(\|f\|_{L_p^*} + \|g\|_{L_p^*})\,.$$

This means that L_p^* is a so called quasi-normed vector space. (In a normed space we have the triangle inequality $\|f+g\| \leqslant \|f\| + \|g\|$, but in a quasi-normed space we only have the quasi-triangle inequality $\|f+g\| \leqslant k(\|f\| + \|g\|)$ for some $k \geqslant 1$.) If $p > 1$ one can, however, as will be seen later on, find a norm on L_p^* and, with this norm, L_p^* becomes a Banach space. One can show that L_1^* is complete but not a normable space. (See Section 1.6.)

The spaces L_p^* are special cases of the more general Lorentz spaces L_{pr}. In their definition we use yet another concept. If f is a μ-measurable function we denote by f^* its *decreasing rearrangement*

$$(4) \qquad f^*(t) = \inf\{\sigma : m(\sigma, f) \leqslant t\}\,.$$

This is a non-negative and non-increasing function on $(0, \infty)$ which is continuous on the right and has the property

$$(5) \qquad m(\rho, f^*) = m(\rho, f)\,, \qquad \rho \geqslant 0\,.$$

(See Figure 2.) Thus f^* is equimeasurable with f. In fact, by (4) we have $f^*(m(\rho, f)) \leqslant \rho$ and thus $m(\rho, f^*) \leqslant m(\rho, f)$. Moreover, since f^* is continuous on the right, $f^*(m(\rho, f^*)) \leqslant \rho$ and hence $m(\rho, f) \leqslant m(\rho, f^*)$.

Note that at all points t where $f^*(t)$ is continuous the relation $\sigma = f^*(t)$ is equivalent to $t = m(\sigma, f)$.

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