# Lectures on Lie Groups

# 李群讲义

W.Y.Hsiang

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# Lectures on Lie Groups

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# CONTENTS

# Lecture 1

# Linear Groups and Linear Representations

## 1. Basic Concepts and Definitions

**Definitions** 1. A topological (resp. Lie) group consists of a group structure and a topological (resp. differentiable) structure such that the multiplication map and the inversion map are continuous (resp. differentiable).

2. A topological (resp. Lie) transformation group consists of a topological (resp. Lie) group $G$, a topological (resp. differentiable) space $X$ and a continuous (resp. differentiable) action map $\Phi$: $G \times X \to X$ satisfying $\Phi(1, x) = x$, $\Phi(g_1, \Phi(g_2, x)) = \Phi(g_1 g_2, x)$.

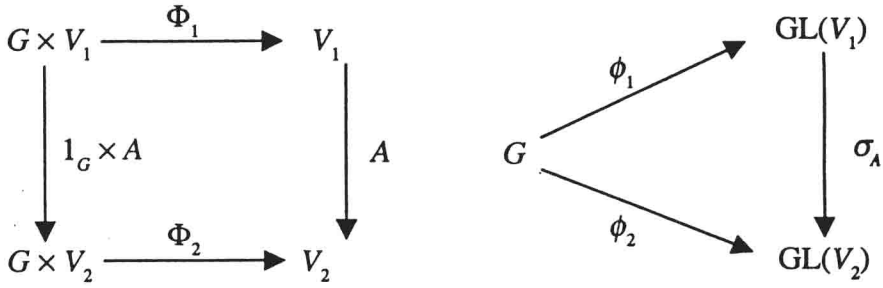
3. If the above space $X$ is a real (resp. complex) vector space and, if for all $g \in G$, the maps $\Phi(g) : X \to X : x \mapsto \Phi(g, x)$ are linear maps, then $G$ is called a real (resp. complex) linear transformation group.

**Notation and Terminology** 1. A space $X$ with a topological (resp. differentiable, linear) transformation of a given group $G$ shall be called a topological (resp. differentiable, linear) $G$-space. In case there is no danger of

ambiguity, we shall always use the simplified notation, $g \cdot x$, to denote $\Phi(g, x)$. In such a multiplicative notation, the defining conditions of the action map $\Phi$ become the familiar forms of $1 \cdot x = x$ and $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.

2. A map $f : X \to Y$ between two $G$-spaces is called a $G$-map if for all $g \in G$ and all $x \in X$, $f(g \cdot x) = g \cdot f(x)$.

3. A linear transformation group $\Phi : G \times V \to V$, or equivalently, a homomorphism $\phi : G \to \mathrm{GL}(V)$, is also called a linear representation of $G$ on $V$. Two linear representations of $G$ on $V_1$ and $V_2$ are said to be equivalent if $V_1$ and $V_2$ are $G$-isomorphic, namely, there exists a linear isomorphism $A$: $V_1 \to V_2$ such that for all $g \in G$ and all $x \in V_1$, $A \cdot \Phi_1(g, x) = \Phi_2(g, Ax)$, or equivalently, one has the following commutative diagrams:




where $\sigma_A(B) = ABA^{-1}$ for $B \in \mathrm{GL}(V_1)$.

4. For a given $G$-space $X$, we shall use $G_x$ to denote the isotropy subgroup of a point $x$ and use $G(x)$ to denote the orbit of $x$, namely

$$G_x = \{g \in G : g \cdot x = x\},$$

$$G(x) = \{g \cdot x : g \in G\}.$$

It is clear that $G_{g \cdot x} = g G_x g^{-1}$ and the map $g \mapsto g \cdot x$ induces a bijection of $G/G_x$ onto $G(x)$.

**Definitions** 1. A (linear) subspace $U$ of a given linear $G$-space $V$ is called an invariant (or $G$-) subspace, if

$$G \cdot U = \{g \cdot x : g \in G, x \in U\} \subseteq U.$$

2. A linear $G$-space $V$ (or its corresponding representation of $G$ on $V$) is said to be irreducible if $\{0\}$ and $V$ are the only invariant subspaces.

3. A linear $G$-space $V$ (or its corresponding representation of $G$ on $V$) is called completely reducible if it can be expressed as the direct sum of *irreducible $G$-subspaces.*

4. The following equations define the induced linear $G$-space structures of two given linear $G$-spaces $V$ and $W$.

(i) direct sum: $V \oplus W$ with $g \cdot (x, y) = (gx, gy)$.
(ii) dual space: $V^*$ with $\langle x, g \cdot x' \rangle = \langle g^{-1} \cdot x, x' \rangle$. (Notice that the inverse in the above definition is needed to ensure that $\langle x, g_1 \cdot (g_2 \cdot x') \rangle = \langle x, (g_1 \cdot g_2) \cdot x' \rangle$ for all $x \in V$, $x' \in V^*$.)
(iii) tensor product: $V \otimes W$ with $g \cdot (x \otimes y) = g \cdot x \otimes g \cdot y$.
(iv) $\mathrm{Hom}(V, W)$: $A \in \mathrm{Hom}(V, W)$, $(g \cdot A)x = gA(g^{-1} \cdot x)$.

It follows from the above definition that the usual canonical isomorphisms such as

$$\mathrm{Hom}(V, W) \cong V^* \otimes W\,,$$

$$(V \otimes W)^* \cong V^* \otimes W^*\,,$$

$$U \otimes (V \oplus W) \cong (U \otimes V) \oplus (U \otimes W)$$

are automatically $G$-isomorphisms. Moreover, an element $A \in \mathrm{Hom}(V, W)$ is a fixed point if and only if it is a $G$-linear map. $g^{-1} \cdot A = A \Leftrightarrow$ for all $g \in G$, $g^{-1}A(gx) = A(x)$, i.e. $A(gx) = gA(x)$.

Of course, one may also define the induced $G$-space structure for the other linear algebra constructions such as $\Lambda^k(V)$, $S^k(V)$, etc., and again the canonical isomorphisms such as $V \otimes V \cong \Lambda^2(V) \oplus S^2(V)$ will also be $G$-isomorphisms.

**Schur Lemma** *Let $V$, $W$ be irreducible (linear) $G$-spaces and $A$: $V \to W$ be a $G$-linear map. Then $A$ is either invertible or $A = 0$.*

**Proof:** Both $\ker A \subseteq V$ and $\mathrm{Im}\, A \subseteq W$ are clearly $G$-subspaces; it follows from the irreducibility assumption that

$$\ker A = \begin{cases} \{0\} \\ V, \end{cases} \qquad \mathrm{Im}\, A = \begin{cases} \{0\} \\ W\,. \end{cases}$$

Therefore, the only possible combinations are exactly either (i) $\ker A = \{0\}$ and $\mathrm{Im}\, A = W$, i.e. $A$ is invertible, or (ii) $\ker A = V$ and $\mathrm{Im}\, A = \{0\}$, i.e. $A = 0$. □

In the special case of $V = W$ and the base field $\mathbb{C}$, one has the following refinement.

**Special Form** *If $V$ is an irreducible $G$-space over $\mathbb{C}$ and $A$ is a $G$-linear self-map of $V$, then $A$ is a scalar multiple, i.e. $A = \lambda_0 \cdot I$ for a suitable $\lambda_0 \in \mathbb{C}$.*

**Proof:** It is obvious that $A - \lambda I$ is also $G$-linear for any $\lambda \in \mathbb{C}$. Let $\lambda_0$ be an eigenvalue of $A$; this exists because $\mathbb{C}$ is algebraically closed. Then $A - \lambda_0 I$ is not invertible and hence must be zero, i.e. $A = \lambda_0 I$. □

**Corollary** *A complex irreducible representation of an Abelian group $G$ is always one-dimensional.*

**Proof:** Let $\phi : G \to \mathrm{GL}(V)$ be a complex irreducible representation. Since $G$ is commutative, $\phi(g) \cdot \phi(g_0) = \phi(g_0) \cdot \phi(g)$ for all $g_0$, $g \in G$. Hence, for each $g$, $\phi(g)$ is a $G$-linear self-map of $V$ and therefore $\phi(g) = \lambda(g) \cdot I$ for a suitable $\lambda(g) \in \mathbb{C}$. $g$, however, is an arbitrary element of $G$, thus $\mathrm{Im}\,\phi = \{\phi(g) : g \in G\} \subset \mathbb{C}^* \cdot I$, the set of nonzero scalar multiples. Therefore any subspace of $V$ is automatically $G$-invariant, and hence it can be irreducible only when $\dim V = 1$. □

## 2. A Brief Overview

Before proceeding to the technical discussion of linear representation theory, let us pause a moment to reflect on some of the special features of linear transformation groups, to think about what are some of the natural problems that one might pursue and to have a brief overview of the fundamental results of such a theory.

Among all kinds of mathematical models, vector space structure is undoubtedly one of the most basic and most useful type; it is a kind of ideal combination of straightforward algebraic operations and simple, natural geometric intuitions. Correspondingly, linear transformation groups also inherit many advantageous nice features. For example, they are conceptually rather elementary and concrete; they are easily accessible to algebraic computations; they can be readily organized by the canonical constructions of linear algebra, e.g., direct sum, tensor product, dual space, etc., and moreover, they also enjoy the beneficial help of geometric interpretation and imagination. Therefore, they are a kind of ideal material to serve as the "films" for taking "reconnaissance pictures".

The theory of representations of groups by linear transformation was created by G. Frobenius, here in Berlin during the years 1896–1903. His basic idea is that one should be able to obtain a rather wholesome understanding of the structure of a given group $G$ by a systematic analysis of the totality of its "linear pictures". Next, let us try to formulate some natural problems along the above lines of thinking.

**1. Problem on complete reducibility**

If all representations of a given group $G$ happen to be automatically completely reducible, then the study of linear $G$-spaces can easily be reduced to that of irreducible ones. Therefore, it is natural to ask "What type of groups have the property that all representations of such groups are automatically completely reducible"?

**2. Problem on irreducibility criterion**

How to determine whether a given representation is irreducible?

**3. Problem on classification**

How to classify irreducible representations of a given group $G$ up to equivalence?

Finally, let us have a preview of some of the remarkable answers to the above basic problems obtained by G. Frobenius and I. Schur.

**Theorem 1.** *If $G$ is a compact topological group, then any real (resp. complex) representation of $G$ is automatically completely reducible.*

The key to the classification theory of linear representations of groups is the following invariant introduced by G. Frobenius.

**Definition** Let $\phi : G \to \mathrm{GL}(V)$ be a given complex representation of $G$. The complex valued function

$$\chi_\phi : G \xrightarrow{\phi} \mathrm{GL}(V) \xrightarrow{\mathrm{tr}} \mathbb{C} : g \mapsto \mathrm{tr}\,\phi(g)$$

is called the character (function) of $\phi$.

1. $\chi_\phi(g) = \mathrm{tr}\,\phi(g)$ is the sum with multiplicities of the eigenvalues of $\phi(g)$. Hence, it is quite obvious that equivalent representations have identical character functions, namely, the character function is an invariant of equivalent classes of representations.

2. If $g_1$, $g_2$ are conjugateplus in $G$, i.e. there is some $g \in G$ such that $g_1 = gg_2g^{-1}$, then

$$\chi_\phi(g_1) = \chi_\phi(gg_2g^{-1}) = \mathrm{tr}(\phi(g)\phi(g_2)\phi(g)^{-1}) = \mathrm{tr}\,\phi(g_2) = \chi_\phi(g_2)\,.$$

Hence, the character function of an arbitrary representation $\phi$ of $G$ has the special property of constancy on each conjugacy class of $G$.

3. If $\psi = \phi_1 \oplus \phi_2$, then it is easy to see that for all $g \in G$,

$$\chi_\psi(g) = \chi_{\phi_1}(g) + \chi_{\phi_2}(g)\,,$$

namely, $\chi_\psi = \chi_{\phi_1} + \chi_{\phi_2}$ as functions.

The most remarkable result of Frobenius–Schur theory is the following classification theorem.

**Theorem 1.** *If $G$ is a compact topological group, then two representations $\phi$ and $\psi$ are equivalent if and only if $\chi_\phi = \chi_\psi$ as functions.*

## 3. Compact Groups, Haar Integral and the Averaging Method

Let $G$ be a finite group and $V$ be a given linear $G$-space. Then, to each point $x \in V$, the center of mass of the orbit $G(x)$ is clearly a fixed point of $V$. Hence, the map

$$x \mapsto \bar{x} = \text{the center of mass of } G(x) = \frac{1}{|G|}\sum_{g\in G} g \cdot x$$

is a canonical projection of $V$ onto $V^G$, the subspace of fixed points in $V$. In terms of a chosen coordinate system, the $i$th coordinate of $\bar{x}$ is simply the average value of the $i$th coordinate of $\{g \cdot x : g \in G\}$. We shall proceed to generalize the above useful method of producing fixed elements, namely, the averaging method, to the general setting of compact topological groups. Of course, the key step is to establish the correct meaning of the average value of a given continuous function $f : G \to \mathbb{R}$.

### 3.1. *Haar integral of functions defined on compact groups*

Let $G$ be a given compact topological group and $C(G)$ be the linear space of all (real valued) continuous functions of $G$ equipped with the sup-norm topology. It is not difficult to show that every continuous function $f \in C(G)$

is automatically uniformly continuous, i.e. to any given $\delta > 0$, there exists a neighborhood $U$ of the identity in $G$ such that $xy^{-1} \in U \Rightarrow |f(x) - f(y)| < \delta$.

The translational transformation of $G \times G$ on $G$, namely,

$$T : (G \times G) \times G \to G : (g_1, g_2) \cdot x \mapsto g_1 x g_2^{-1}$$

naturally induces a continuous linear transformation of $G \times G$ on $C(G)$, namely,

$$[(g_1, g_2) \cdot f](x) = f(g_1^{-1} x g_2), \qquad f \in C(G), \qquad (g_1, g_2) \in G \times G.$$

**Theorem 3.** *There exists a unique $G$-projection $I : C(G) \to \mathbb{R}$ (the subspace of constant functions). [$I(f)$ is called the average value, or Haar integral, of $f$.]*

**Proof:** (a sketch) i. Let $A$ be a finite subset of $G \times G$ with multiplicities and $f \in C(G)$. Set $\Gamma(A, f)$ to be the center of mass of $A \cdot f$, namely,

$$\Gamma(A, f) = \frac{1}{|A|} \sum_{a \in A} m(a) \cdot a \cdot f,$$

where $m(a)$ is the multiplicity of $a$ and $|A| = \sum m(a)$ is the total weight. For two finite subsets $A$, $B$ of $G \times G$ with multiplicities, $A \cdot B$ is again a finite subset with multiplicities and it is easy to check that $\Gamma(A, \Gamma(B, f)) = \Gamma(A \cdot B, f)$.

ii. Set $\Delta_f = \{\Gamma(A, f) : A$ is a finite subset with multiplicities of $G \times G\}$. For each $h \in C(G)$, set

$$\omega(h) = \max\{h(x) : x \in G\} - \min\{h(x) : x \in G\}.$$

Let $C(f)$ be the greatest lower bound of $\{\omega(h) : h \in \Delta_f\}$ and $\{h_n\}$ be a minimizing sequence, namely, $\omega(h_n) \to C(f)$ as $n \to \infty$. It is straightforward to check that $\Delta_f$ is a family of equicontinuous functions, namely, to any given $\delta > 0$, there exists a neighborhood $U$ of the identity in $G$ such that

$$(\forall h \in \Delta_f) xy^{-1} \in U \Rightarrow |h(x) - h(y)| < \delta.$$

Therefore, there exists a converging subsequence of $\{h_n\}$ and hence one may assume that $\{h_n\}$ is itself convergent to begin with.

iii. Set $\bar{h} = \lim h_n$. Then it is clear that $\omega(\bar{h}) = C(f)$. Finally, one proves by contradiction that $\omega(\bar{h}) = C(f) = 0$! For otherwise, one can always choose a suitable finite subset $A \in G \times G$ such that

$$\omega(\Gamma(A, \bar{h})) < \omega(\bar{h}) = C(f).$$

Moreover, it is straightforward to check that $\Gamma(A, h_n)$ converges to $\Gamma(A, \bar{h})$ and $\lim \omega(\Gamma(A, h_n)) = \omega(\Gamma(A, \bar{h}))$. But all $\Gamma(A, h_n)$ are obviously also element of $\Delta_f$, which contradicts the fact that $C(f)$ is the greatest lower bound for all $\omega(h)$. (We refer to L. S. Pontriagin's book *Topological Groups* for the details of the above proof due to von Neumann.) □

The above continuous $G \times G$-equivariant, linear functional $I : C(G) \to \mathbb{R}$ uniquely determines a $G \times G$-invariant measure of total measure 1 on $G$ (called the Haar measure) such that $I(f) = \int_G f(g)dg$ for all $f \in C(G)$.

### 3.2. *Existence of invariant inner (resp. Hermitian) product*

As the first application of the averaging method, let us establish the following basic fact which includes Theorem 1 as an easy corollary.

**Theorem 4.** *Let $V$ be a given real (resp. complex) linear $G$-space. If $G$ is a compact topological groups, then there exists a $G$-invariant inner (resp. Hermitian) product on $V$, namely*

$$(g \cdot x, g \cdot y) = (x, y) \quad \text{for all } x, y \in V, g \in G$$

**Proof:** Let $\langle x, y \rangle$ be an arbitrary inner (resp. Hermitian) product on $V$. Set

$$(x, y) = \int_G \langle g \cdot x, g \cdot y \rangle dg \,.$$

It is straightforward to verify that $(x, y)$ is again an inner (resp. Hermitian) product on $V$, and moreover

$$(a \cdot x, a \cdot y) = \int_G \langle ga \cdot x, ga \cdot y \rangle dg \,.$$

Letting $g' = ga$, $dg' = dg$, then

$$(a \cdot x, a \cdot y) = \int_G \langle g' \cdot x, g' \cdot y \rangle dg' = (x, y) \,.$$

□

**Definition** A real (resp. complex) linear $G$-space with an invariant inner (resp. Hermitian) product is called an orthogonal (resp. unitary) $G$-space, and the corresponding representation is called an orthogonal (resp. unitary) representation.

In an orthogonal (resp. unitary) $G$-space $V$, the perpendicular subspace to an *invariant* subspace is automatically also an invariant subspace.

**Proof of Theorem 1:** By Theorem 4, one may equip $V$ with an invariant inner (resp. Hermitian) product. Let $U$ by a positive dimensional irreducible sub-$G$-space of $V$ and $U^\perp$ be its perpendicular subspace. Then $V = U \oplus U^\perp$ is a decomposition of $V$ into the direct sum of sub-$G$-spaces, $\dim U^\perp < \dim V$. From here, the proof of Theorem 1 follows by a simple induction on $\dim V$. □

**Exercises** 1. Let $O(n) \subset \mathrm{GL}(n, \mathbb{R})$ (resp. $U(n) \subset \mathrm{GL}(n, \mathbb{C})$) be the subgroup of orthogonal (resp. unitary) matrices. Show that they are compact.

2. Let $G \subset \mathrm{GL}(n, \mathbb{R})$ (resp. $\mathrm{GL}(n, \mathbb{C})$) be a compact subgroup. Show that there exists a suitable element $A$ in $\mathrm{GL}(n, \mathbb{R})$ (resp. $\mathrm{GL}(n, \mathbb{C})$) such that

$$AGA^{-1} \subset O(n) \quad (\text{resp. } U(n)) .$$

3. Show that $O(n)$ (resp. $U(n)$) is a *maximal* compact subgroup of $\mathrm{GL}(n, \mathbb{R})$ (resp. $\mathrm{GL}(n, \mathbb{C})$) and any two maximal compact subgroups of $\mathrm{GL}(n, \mathbb{R})$ (resp. $\mathrm{GL}(n, \mathbb{C})$) must be mutually conjugate.

4. Let $\phi$, $\psi$ be two complex representations of a compact group $G$. Then $\chi_{\phi\otimes\psi}(g) = \chi_\phi(g) \cdot \chi_\psi(g)$ for all $g \in G$. (Thanks to Theorem 4, $\phi(g)$ and $\psi(g)$ are always diagonalizable.)

## 4. Frobenius–Schur Orthogonality and the Character Theory

Now let us apply the averaging method to analyze the deep implications of the Schur lemma.

Case 1: Let $\phi : G \to \mathrm{GL}(V)$, $\psi : G \to \mathrm{GL}(W)$ be two *non-equivalent, irreducible* complex representations of a compact group $G$. Then it follows from the Schur lemma that

$$\mathrm{Hom}_G(V, W) = \mathrm{Hom}(V, W)^G = \{0\} .$$

(Recall that $X^G$ denotes the fixed point set of a $G$-space $X$.) Therefore, it follows from the averaging method that for all $A \in \mathrm{Hom}(V, W)$

$$\int_G g \cdot A \, dg = \int_G \psi(g) \cdot A \cdot \phi(g)^{-1} dg = 0 ,$$

because $\int_G g \cdot A\, dg$ is the center of mass of $G(A)$ and, of course, it is always a fixed point!

By Theorem 4, one may equip both $V$ and $W$ with invariant Hermitian products and compute the above powerful equation in its matrix form with respect to chosen orthonormal bases in $V$ and $W$. Let $E_{ik}$ be the linear map which maps the $k$th base vector of $V$ to the $i$th base vector of $W$ and all the other base vectors of $V$ to zero.

Since the above equation is linear with respect to the parameter $A$ and $\{E_{ik} : 1 \leq i \leq \dim W, 1 \leq k \leq \dim V\}$ already forms a basis of $\mathrm{Hom}(V, W)$, one needs only to compute the special cases of $A = E_{ik}$. Set

$$\phi(g) = (\phi_{kl}(g)), \qquad \psi(g) = (\psi_{ij}(g)).$$

($\phi_{kl}(g), \psi_{ij}(g) \in C(G)$ are called representation functions.) One has

$$\begin{aligned} 0 &= \int_G g \cdot E_{ab} dg = \int_G (\psi_{ij}(g)) \cdot E_{ab} \cdot (\bar{\phi}_{kl}(g))^t dg \\ &= \int_G (\psi_{ia}(g) \cdot \bar{\phi}_{kb}(g)) dg . \end{aligned}$$

Hence

$$\int_G \psi_{ia}(g) \cdot \bar{\phi}_{kb}(g) dg = 0 ,$$

for $1 \leq i, a \leq \dim W$, $1 \leq k, b \leq \dim V$.

Case 2: The special form of the Schur lemma asserts that

$$\mathrm{Hom}_G(V, V) = \mathrm{Hom}(V, V)^G = \{\lambda \cdot I : \lambda \in C^*\} .$$

Hence, it again follows from the averaging method that

$$\int_G g \cdot B dg = \int_G \phi(g) \cdot B \cdot \phi(g)^{-1} dg = \lambda_B \cdot I ,$$

where $\lambda_B$ is a yet-to-be-determined complex number solely depending on $B$. Exploiting the linearity and the conjugate invariance of the trace, one has

$$\begin{aligned} \lambda_B \cdot \dim V &= \mathrm{tr}\, \lambda_B \cdot I = \mathrm{tr} \int_G \phi(g) \cdot B \cdot \phi(g)^{-1} dg \\ &= \int_G \mathrm{tr}(\phi(g) \cdot B \cdot \phi(g)^{-1}) dg = \int_G \mathrm{tr}\, B dg \\ &= \mathrm{tr}\, B \end{aligned}$$

which determines the value of $\lambda_B$, namely