Universitext

Haim Brezis

Functional Analysis, Sobolev Spaces and Partial Differential Equations

泛函分析、索伯列夫空间和偏微分方程

Springer

世界图书出版公司
www.wpcbj.com.cn

Haim Brezis

Functional Analysis, Sobolev Spaces and Partial Differential Equations

Springer

图书在版编目(CIP)数据

泛函分析、索伯列夫空间和偏微分方程 = Functional Analysis, Sobolev Spaces and Partial Differential Equations:英文/(美)布勒齐(Brezis, H.)著.—影印本.—北京:世界图书出版公司北京公司,2015.7

ISBN 978-7-5100-9677-8

Ⅰ.①泛…　Ⅱ.①布…　Ⅲ.①泛函分析—英文②索伯列夫空间—英文③偏微分方程—英文　Ⅳ.①O177 ②O175.2

中国版本图书馆 CIP 数据核字(2015)第 152955 号

Functional Analysis, Sobolev Spaces and Partial Differential Equations

泛函分析、索伯列夫空间和偏微分方程

著　　者: Haim Brezis
责任编辑: 刘　慧　岳利青
装帧设计: 任志远

出版发行: 世界图书出版公司北京公司
地　　址: 北京市东城区朝内大街 137 号
邮　　编: 100010
电　　话: 010-64038355(发行)　64015580(客服)　64033507(总编室)
网　　址: http://www.wpcbj.com.cn
邮　　箱: wpcbjst@vip.163.com
销　　售: 新华书店
印　　刷: 三河市国英印务有限公司
开　　本: 711mm×1245mm　1/24
印　　张: 26
字　　数: 500 千
版　　次: 2015 年 7 月第 1 版　　2015 年 7 月第 1 次印刷
版权登记: 01-2015-2537
ISBN 978-7-5100-9677-8　　　**定价:** 98.00 元

Universitext

For other titles in this series, go to
www.springer.com/series/223

Haim Brezis
Distinguished Professor
Department of Mathematics
Rutgers University
Piscataway, NJ 08854
USA
brezis@math.rutgers.edu

and

Professeur émérite, Université Pierre et Marie Curie (Paris 6)

and

Visiting Distinguished Professor at the Technion

ISBN 978-0-387-70913-0 e-ISBN 978-0-387-70914-7
DOI 10.1007/978-0-387-70914-7
Springer New York Dordrecht Heidelberg London

Library of Congress Control Number: 2010938382

Mathematics Subject Classification (2010): 35Rxx, 46Sxx, 47Sxx

To Felix Browder, a mentor and close friend, who taught me to enjoy PDEs through the eyes of a functional analyst

Preface

This book has its roots in a course I taught for many years at the University of Paris. It is intended for students who have a good background in real analysis (as expounded, for instance, in the textbooks of G. B. Folland [2], A. W. Knapp [1], and H. L. Royden [1]). I conceived a program mixing elements from two distinct "worlds": functional analysis (FA) and partial differential equations (PDEs). The first part deals with abstract results in FA and operator theory. The second part concerns the study of spaces of functions (of one or more real variables) having specific differentiability properties: the celebrated Sobolev spaces, which lie at the heart of the modern theory of PDEs. I show how the abstract results from FA can be applied to solve PDEs. The Sobolev spaces occur in a wide range of questions, in both pure and applied mathematics. They appear in linear and nonlinear PDEs that arise, for example, in differential geometry, harmonic analysis, engineering, mechanics, and physics. They belong to the toolbox of any graduate student in analysis.

Unfortunately, FA and PDEs are often taught in separate courses, even though they are intimately connected. Many questions tackled in FA originated in PDEs (for a historical perspective, see, e.g., J. Dieudonné [1] and H. Brezis–F. Browder [1]). There is an abundance of books (even voluminous treatises) devoted to FA. There are also numerous textbooks dealing with PDEs. However, a synthetic presentation intended for graduate students is rare. and I have tried to fill this gap. Students who are often fascinated by the most abstract constructions in mathematics are usually attracted by the elegance of FA. On the other hand, they are repelled by the never-ending PDE formulas with their countless subscripts. I have attempted to present a "smooth" transition from FA to PDEs by analyzing first the simple case of one-dimensional PDEs (i.e., ODEs—ordinary differential equations), which looks much more manageable to the beginner. In this approach, I expound techniques that are possibly too sophisticated for ODEs, but which later become the cornerstones of the PDE theory. This layout makes it much easier for students to tackle elaborate higher-dimensional PDEs afterward.

A previous version of this book, originally published in 1983 in French and followed by numerous translations, became very popular worldwide, and was adopted as a textbook in many European universities. A deficiency of the French text was the lack of exercises. The present book contains a wealth of problems. I plan to add even more in future editions. I have also outlined some recent developments, especially in the direction of nonlinear PDEs.

Brief user's guide

1. Statements or paragraphs preceded by the bullet symbol • **are extremely important**, and it is essential to grasp them well in order to understand what comes afterward.
2. Results marked by the star symbol ⋆ **can be skipped** by the beginner; they are of interest only to advanced readers.
3. In each chapter I have labeled propositions, theorems, and corollaries in a continuous manner (e.g., Proposition 3.6 is followed by Theorem 3.7, Corollary 3.8, etc.). Only the remarks and the lemmas are numbered separately.
4. In order to simplify the presentation I assume that all vector spaces are over $\mathbb{R}$. Most of the results remain valid for vector spaces over $\mathbb{C}$. I have added in Chapter 11 a short section describing similarities and differences.
5. Many chapters are followed by numerous exercises. Partial solutions are presented at the end of the book. More elaborate problems are proposed in a separate section called "Problems" followed by "Partial Solutions of the Problems." The problems usually require knowledge of material coming from various chapters. I have indicated at the beginning of each problem which chapters are involved. Some exercises and problems expound results stated without details or without proofs in the body of the chapter.

Acknowledgments

During the preparation of this book I received much encouragement from two dear friends and former colleagues: Ph. Ciarlet and H. Berestycki. I am very grateful to G. Tronel, M. Comte, Th. Gallouet, S. Guerre-Delabrière, O. Kavian, S. Kichenassamy, and the late Th. Lachand-Robert, who shared their "field experience" in dealing with students. S. Antman, D. Kinderlehrer, and Y. Li explained to me the background and "taste" of American students. C. Jones kindly communicated to me an English translation that he had prepared for his personal use of some chapters of the original French book. I owe thanks to A. Ponce, H.-M. Nguyen, H. Castro, and H. Wang, who checked carefully parts of the book. I was blessed with two extraordinary assistants who typed most of this book at Rutgers: Barbara Miller, who is retired, and now Barbara Mastrian. I do not have enough words of praise and gratitude for their constant dedication and their professional help. They always found attractive solutions to the challenging intricacies of PDE formulas. Without their enthusiasm and patience this book would never have been finished. It has been a great pleasure, as

ever, to work with Ann Kostant at Springer on this project. I have had many opportunities in the past to appreciate her long-standing commitment to the mathematical community.

The author is partially supported by NSF Grant DMS-0802958.

Haim Brezis
Rutgers University
March 2010

Contents

Chapter 1
The Hahn–Banach Theorems. Introduction to the Theory of Conjugate Convex Functions

1.1 The Analytic Form of the Hahn–Banach Theorem: Extension of Linear Functionals

Let E be a vector space over $\mathbb{R}$. We recall that a *functional* is a function defined on E, or on some subspace of E, *with values in* $\mathbb{R}$. The main result of this section concerns the extension of a linear functional defined on a linear subspace of E by a linear functional defined on all of E.

Theorem 1.1 (Helly, Hahn–Banach analytic form). *Let* $p : E \to \mathbb{R}$ *be a function satisfying*[1]

$$(1) \qquad p(\lambda x) = \lambda p(x) \quad \forall x \in E \quad \text{and} \quad \forall \lambda > 0,$$

$$(2) \qquad p(x + y) \le p(x) + p(y) \quad \forall x, y \in E.$$

Let $G \subset E$ *be a linear subspace and let* $g : G \to \mathbb{R}$ *be a linear functional such that*

$$(3) \qquad g(x) \le p(x) \quad \forall x \in G.$$

Under these assumptions, there exists a linear functional f *defined on all of* E *that extends* g, *i.e.,* $g(x) = f(x)\ \forall x \in G$, *and such that*

$$(4) \qquad f(x) \le p(x) \quad \forall x \in E.$$

The proof of Theorem 1.1 depends on Zorn's lemma, which is a celebrated and very useful property of ordered sets. Before stating Zorn's lemma we must clarify some notions. Let P be a set with a (partial) order relation $\le$. We say that a subset $Q \subset P$ is *totally ordered* if for any pair (a, b) in Q either $a \le b$ or $b \le a$ (or both!). Let $Q \subset P$ be a subset of P; we say that $c \in P$ is an *upper bound* for Q if $a \le c$ for every $a \in Q$. We say that $m \in P$ is a *maximal* element of P if there is *no* element

[1] A function p satisfying (1) and (2) is sometimes called a *Minkowski functional*.

H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*,
DOI 10.1007/978-0-387-70914-7_1,

$x \in P$ such that $m \leq x$, except for $x = m$. Note that a maximal element of P need not be an upper bound for P.

We say that P is *inductive* if every totally ordered subset Q in P has an upper bound.

• **Lemma 1.1 (Zorn).** *Every nonempty ordered set that is inductive has a maximal element.*

Zorn's lemma follows from the axiom of choice, but we shall not discuss its derivation here; see, e.g., J. Dugundji [1], N. Dunford–J. T. Schwartz [1] (Volume 1, Theorem 1.2.7), E. Hewitt–K. Stromberg [1], S. Lang [1], and A. Knapp [1].

Remark 1. Zorn's lemma has many important applications in analysis. It is a *basic tool* in proving some *seemingly innocent existence statements* such as "every vector space has a basis" (see Exercise 1.5) and "on any vector space there are nontrivial linear functionals." Most analysts do not know how to prove Zorn's lemma; but it is quite essential for an analyst to understand the statement of Zorn's lemma and to be able to use it properly!

Proof of Lemma 1.2. Consider the set

$$P = \left\{ h : D(h) \subset E \to \mathbb{R} \;\middle|\; \begin{array}{l} D(h) \text{ is a linear subspace of } E, \\ h \text{ is linear, } G \subset D(h), \\ h \text{ extends } g, \text{ and } h(x) \leq p(x) \quad \forall x \in D(h) \end{array} \right\}.$$

On P we define the order relation

$$(h_1 \leq h_2) \Leftrightarrow (D(h_1) \subset D(h_2) \text{ and } h_2 \text{ extends } h_1).$$

It is clear that P is nonempty, since $g \in P$. We claim that P is *inductive*. Indeed, let $Q \subset P$ be a totally ordered subset; we write Q as $Q = (h_i)_{i \in I}$ and we set

$$D(h) = \bigcup_{i \in I} D(h_i), \quad h(x) = h_i(x) \quad \text{if } x \in D(h_i) \text{ for some } i.$$

It is easy to see that the definition of h makes sense, that $h \in P$, and that h is an upper bound for Q. We may therefore apply Zorn's lemma, and so we have a maximal element f in P. We claim that $D(f) = E$, which completes the proof of Theorem 1.1.

Suppose, by contradiction, that $D(f) \neq E$. Let $x_0 \notin D(f)$; set $D(h) = D(f) + \mathbb{R}x_0$, and for every $x \in D(f)$, set $h(x + tx_0) = f(x) + t\alpha$ ($t \in \mathbb{R}$), where the constant $\alpha \in \mathbb{R}$ will be chosen in such a way that $h \in P$. We must ensure that

$$f(x) + t\alpha \leq p(x + tx_0) \quad \forall x \in D(f) \quad \text{and} \quad \forall t \in \mathbb{R}.$$

In view of (1) it suffices to check that

$$\begin{cases} f(x) + \alpha \le p(x + x_0) & \forall x \in D(f), \\ f(x) - \alpha \le p(x - x_0) & \forall x \in D(f). \end{cases}$$

In other words, we must find some α satisfying

$$\sup_{y \in D(f)} \{f(y) - p(y - x_0)\} \le \alpha \le \inf_{x \in D(f)} \{p(x + x_0) - f(x)\}.$$

Such an α exists, since

$$f(y) - p(y - x_0) \le p(x + x_0) - f(x) \quad \forall x \in D(f), \quad \forall y \in D(f);$$

indeed, it follows from (2) that

$$f(x) + f(y) \le p(x + y) \le p(x + x_0) + p(y - x_0).$$

We conclude that $f \le h$; but this is impossible, since f is maximal and $h \ne f$.

We now describe some simple applications of Theorem 1.1 to the case in which E is a *normed vector space* (n.v.s.) with norm $\| \ \|$.

Notation. We denote by $E^\star$ the *dual space* of E, that is, the space of all *continuous linear functionals on* E; the (dual) *norm on* $E^\star$ is defined by

$$\|f\|_{E^\star} = \sup_{\substack{\|x\| \le 1 \\ x \in E}} |f(x)| = \sup_{\substack{\|x\| \le 1 \\ x \in E}} f(x). \tag{5}$$

When there is no confusion we shall also write $\|f\|$ instead of $\|f\|_{E^\star}$.

Given $f \in E^\star$ and $x \in E$ we shall often write $\langle f, x \rangle$ instead of $f(x)$; we say that $\langle \ , \ \rangle$ is the *scalar product for the duality* $E^\star, E$.

It is well known that $E^\star$ is a Banach space, i.e., $E^\star$ is complete (even if E is not); this follows from the fact that $\mathbb{R}$ is complete.

• **Corollary 1.2.** *Let $G \subset E$ be a linear subspace. If $g : G \to \mathbb{R}$ is a continuous linear functional, then there exists $f \in E^\star$ that extends g and such that*

$$\|f\|_{E^\star} = \sup_{\substack{x \in G \\ \|x\| \le 1}} |g(x)| = \|g\|_{G^\star}.$$

Proof. Use Theorem 1.1 with $p(x) = \|g\|_{G^\star} \|x\|$.

• **Corollary 1.3.** *For every $x_0 \in E$ there exists $f_0 \in E^\star$ such that*

$$\|f_0\| = \|x_0\| \text{ and } \langle f_0, x_0 \rangle = \|x_0\|^2.$$

Proof. Use Corollary 1.2 with $G = \mathbb{R}x_0$ and $g(tx_0) = t\|x_0\|^2$, so that $\|g\|_{G^\star} = \|x_0\|$.

Remark 2. The element f_0 given by Corollary 1.3 is in general not unique (try to construct an example or see Exercise 1.2). However, if $E^\star$ is strictly convex[2]—for example if E is a Hilbert space (see Chapter 5) or if $E = L^p(\Omega)$ with $1 < p < \infty$ (see Chapter 4)—then f_0 is unique. In general, we set, for every $x_0 \in E$,

$$F(x_0) = \left\{ f_0 \in E^\star;\ \|f_0\| = \|x_0\| \text{ and } \langle f_0, x_0 \rangle = \|x_0\|^2 \right\}.$$

The (multivalued) map $x_0 \mapsto F(x_0)$ is called the *duality map* from E into $E^\star$; some of its properties are described in Exercises 1.1, 1.2, and 3.28 and Problem 13.

• **Corollary 1.4.** *For every $x \in E$ we have*

$$\|x\| = \sup_{\substack{f \in E^\star \\ \|f\| \leq 1}} |\langle f, x \rangle| = \max_{\substack{f \in E^\star \\ \|f\| \leq 1}} |\langle f, x \rangle|. \tag{6}$$

Proof. We may always assume that $x \neq 0$. It is clear that

$$\sup_{\substack{f \in E^\star \\ \|f\| \leq 1}} |\langle f, x \rangle| \leq \|x\|.$$

On the other hand, we know from Corollary 1.3 that there is some $f_0 \in E^\star$ such that $\|f_0\| = \|x\|$ and $\langle f_0, x \rangle = \|x\|^2$. Set $f_1 = f_0/\|x\|$, so that $\|f_1\| = 1$ and $\langle f_1, x \rangle = \|x\|$.

Remark 3. Formula (5)—which is a *definition*—should not be confused with formula (6), which is a *statement*. In general, the "sup" in (5) is *not achieved*; see, e.g., Exercise 1.3. However, the "sup" in (5) is achieved if E is a reflexive Banach space (see Chapter 3); a deep result due to R. C. James asserts the converse: if E is a Banach space such that for every $f \in E^\star$ the sup in (5) is achieved, then E is reflexive; see, e.g., J. Diestel [1, Chapter 1] or R. Holmes [1].

1.2 The Geometric Forms of the Hahn–Banach Theorem: Separation of Convex Sets

We start with some preliminary facts about hyperplanes. In the following, E denotes an n.v.s.

Definition. An affine *hyperplane* is a subset H of E of the form

$$H = \{x \in E\,;\ f(x) = \alpha\},$$

where f is a linear functional[3] that does not vanish identically and $\alpha \in \mathbb{R}$ is a given constant. We write $H = [f = \alpha]$ and say that $f = \alpha$ is the equation of H.

[2] A normed space is said to be *strictly convex* if $\|tx + (1-t)y\| < 1$, $\forall t \in (0,1)$, $\forall x, y$ with $\|x\| = \|y\| = 1$ and $x \neq y$; see Exercise 1.26.

[3] We do not assume that f is continuous (in every infinite-dimensional normed space there exist discontinuous linear functionals; see Exercise 1.5).

Proposition 1.5. *The hyperplane $H = [f = \alpha]$ is closed if and only if f is continuous.*

Proof. It is clear that if f is continuous then H is closed. Conversely, let us assume that H is closed. The complement H^c of H is open and nonempty (since f does not vanish identically). Let $x_0 \in H^c$, so that $f(x_0) \neq \alpha$, for example, $f(x_0) < \alpha$.

Fix $r > 0$ such that $B(x_0, r) \subset H^c$, where

$$B(x_0, r) = \{x \in E\,;\, \|x - x_0\| < r\}.$$

We claim that

$$f(x) < \alpha \quad \forall x \in B(x_0, r). \tag{7}$$

Indeed, suppose by contradiction that $f(x_1) > \alpha$ for some $x_1 \in B(x_0, r)$. The segment

$$\{x_t = (1-t)x_0 + tx_1\,;\, t \in [0, 1]\}$$

is contained in $B(x_0, r)$ and thus $f(x_t) \neq \alpha$, $\forall t \in [0, 1]$; on the other hand, $f(x_t) = \alpha$ for some $t \in [0, 1]$, namely $t = \frac{f(x_1)-\alpha}{f(x_1)-f(x_0)}$, a contradiction, and thus (7) is proved. It follows from (7) that

$$f(x_0 + rz) < \alpha \quad \forall z \in B(0, 1).$$

Consequently, f is continuous and $\|f\| \leq \frac{1}{r}(\alpha - f(x_0))$.

Definition. Let A and B be two subsets of E. We say that the hyperplane $H = [f = \alpha]$ *separates* A *and* B if

$$\boxed{f(x) \leq \alpha \quad \forall x \in A \quad \text{and} \quad f(x) \geq \alpha \quad \forall x \in B.}$$

We say that H *strictly separates* A and B if there exists some $\varepsilon > 0$ such that

$$\boxed{f(x) \leq \alpha - \varepsilon \quad \forall x \in A \text{ and } f(x) \geq \alpha + \varepsilon \quad \forall x \in B.}$$

Geometrically, the separation means that A lies in one of the half-spaces determined by H, and B lies in the other; see Figure 1.

Finally, we recall that a subset $A \subset E$ is *convex* if

$$\boxed{tx + (1-t)y \in A \quad \forall x, y \in A, \quad \forall t \in [0, 1].}$$

• **Theorem 1.6 (Hahn–Banach, first geometric form).** *Let $A \subset E$ and $B \subset E$ be two nonempty convex subsets such that $A \cap B = \emptyset$. Assume that one of them is open. Then there exists a closed hyperplane that separates A and B.*

The proof of Theorem 1.6 relies on the following two lemmas.

Lemma 1.2. *Let $C \subset E$ be an open convex set with $0 \in C$. For every $x \in E$ set*