A Study on Eigenvalues of Higher-Order Tensors and Related Polynomial Optimization Problems

（高阶张量特征值和相关多项式优化问题研究）

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To the first author's parents:
Yang Cunliang and Huang Linlin
And wife:
Chen Yiying

Preface

In this book, we study the eigenvalues problems of higher order tensors and several specific polynomial optimization problems.

The concepts of eigenvalues and eigenvectors of higher order tensors were presented in 2005 by Qi and Lim independently. Since then, more and more scholars have devoted to this new research field and have made great progress in the past decade. Particularly the various properties of H-eigenvalues of nonnegative tensors and solving methods have been developed. In the first part of this book we systematically introduce related contents. Problems on finding the maximal or minimal H-eigenvalues and Z-eigenvalues are special class of polynomial optimization problems, while polynomial optimization is attracting great attention due to its importance and has been witnessed rapid development in the past years. It is well known that in general a polynomial optimization problem is nonconvex and NP-hard. So people attempt to use suitable relaxed convex optimization problem to approximate a concrete nonconvex optimization problem. Semidefinite programming (SDP) is an important branch of convex optimization and has been studied extensively in theory as well as in algorithms in the past two decades. Besides it is important in itself right, SDP has been found to be very useful as the convex approximation of nonconvex quadratic programming and some discrete optimization problems. In the second part of this book we devote to studying several special polynomial optimization problems by using SDP relaxation strategy, such as maximal real tensor eigenvalue problem and their extended problems.

Most of the contents of this book are composed of PhD dissertation of the first author, which have been published in journals.

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Yang Qingzhi
Yang Yuning
Nov. 2014

Notations

$x, y, \cdots$	Vectors
$A, B, \cdots$	Matrices
$\mathcal{A}, \mathcal{B}, \cdots$	Tensors
$\mathbb{R}, \mathbb{C}, \mathbb{Q}$	Real, complex and rational number field
$\mathbb{R}^n, \mathbb{R}^n_+, \mathbb{R}^n_{++}$	n-dimensional real (nonnegative, positive) vector space
$\mathbb{R}^{n\times n}, \mathbb{R}^{n\times n}_+, \mathbb{R}^{n\times n}_{++}$	$n \times n$ dimensional real (nonnegative, positive) matrix space
$S^{n\times n}, S^{n\times n}_+, S^{n\times n}_{++}$	$n \times n$ real symmetric (positive semidefinite, positive definite) matrix space
$\\|\cdot\\|$	ℓ_2 norm of a vector
$\\|\cdot\\|_F$	Frobenius norm of a matrix or a tensor
T	Vector or matrix transpose
rank$(\cdot)$	Rank of a matrix
trace$(\cdot)$	Trace of a marix or a tensor
$\bullet$	Matrix inner product
$\circ$	Hadamard operator
$\otimes$	Kronecker product
$\times_d$	Tensor-matrix mode-d product
$\vert\cdot\vert$	Entry-wise absolute values of a vector, a matrix or a tensor
diag(X)	The diagonal entries of a matrix X
Diag(x)	A matrix whose diagonal entries are the vector x
$\boldsymbol{M}(\cdot)$	Tensor-matrix unfolding operator
$\boldsymbol{T}(\cdot)$	Matrix-tensor folding operator
$\boldsymbol{V}(\cdot)$	Matrix-vector unfolding operator
$\boldsymbol{L}(\cdot)$	Vector-matrix folding operator

Contents

Chapter 1

Introduction

Tensor is a hot topic in the past decade. Nowadays, many real world problems can be modeled as tensor problems, just to name a few: signal processing[36, 101], data analysis[17, 32], chemometrics[12, 13, 111], hypergraph theory[34, 66], diffusion magnetic resonance imaging (MRI) [3, 6, 44], quantum entanglement in quantum physics [35], higher order Markov chains[107] and elastic materials analysis[50, 77]. Specifically, a tensor can be viewed as a multiarray: if a vector $a = [a_1, \cdots, a_n]^{\mathrm{T}} \in \mathbb{R}^n$ is a one-way-array, a matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a two-way-array, then a tensor of order m dimension n: $\mathcal{A} = (a_{i_1\cdots i_n})$ is a multiarray. Here "order" refers to the number of indices of each entry of $\mathcal{A}$, e.g., a matrix is a tensor of order 2. Just imagine we have l matrices of size $m \times n$. By stacking them up together, we get a cube, which is a $l \times m \times n$ tensor. If we merge k such tensors together, then we get a hyper-cube, which is a $k \times l \times m \times n$ tensor, and so on. Particularly, if $k = l = m = n = \cdots$, i.e., all the dimensions of a tensor are the same, then the tensor is called a "square" tensor, which generalizes the "square" matrix and is an important type of tensors studied in this book.

We are particularly interested in the (spectral) properties of eigenvalue problems of higher order tensor and related polynomial optimization problems. In this book, the main concerns are the following five topics:

- Spectral properties and algorithms of H-eigenvalue problems;
- Spectral properties and algorithms of H-singular value problems;
- Properties and algorithms of Z-eigenvalue problems;
- Approximation methods of biquadratic optimization problems;
- Approximation methods of trilinear optimization problems.

We will introduce the related concepts and problems in the rest of this chapter.

1.1 Eigenvalues problems of higher order tensors

It is well known that eigenvalues play an important role in matrix theory. For a matrix $A \in \mathbb{C}^{n\times n}$, if there exists a pair $(\lambda, x) \in \mathbb{C} \times \mathbb{C}^n$ with $x \neq 0$ such that

$$Ax = \lambda x,$$

then λ is called an eigenvalue of A, and x is an eigenvector corresponding to λ. To study tensor problems, one may naturally ask a question: can we define eigenvalues and eigenvectors on tensors? The answer is positive. In 2005, the concept of eigenvalues and eigenvectors of a symmetric tensor with order even was introduced by Qi[119].

To be more specific, let $\mathcal{A} = (a_{i_1 \cdots i_m})$ be an order m n-th dimensional real square tensor. If there is a complex number λ and a nonzero complex vector x that are solutions of the following homogeneous polynomial equations:

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}, \tag{1.1}$$

then λ is called an eigenvalue of $\mathcal{A}$ and x the eigenvector of $\mathcal{A}$ associated with λ. In problem (1.1), $\mathcal{A}x^{m-1}$ and $x^{[m-1]}$ are vectors, whose i-th entries are given by

$$\begin{aligned}(\mathcal{A}x^{m-1})_i &= \sum_{i_2,\cdots,i_m=1}^{n} a_{i,i_2\cdots i_m} x_{i_2} \cdots x_{i_m},\\ (x^{[m-1]})_i &= x_i^{m-1},\end{aligned}$$

respectively. If λ and x are restricted in the real field, then (λ, x) is called an H-eigenpaire. If an eigenvalue is not an H-eigenvalue, we call it an N-eigenvalue of $\mathcal{A}$. Besides the H-eigenvalues, Qi defined the Z-eigenvalues[119]: a real number λ and a real vector x are called Z-eigenvalue of $\mathcal{A}$ and a Z-eigenvector of $\mathcal{A}$ associated with the Z-eigenvalue λ respectively, if they are solutions of the following system:

$$\begin{cases} \mathcal{A}x^{m-1} = \lambda x, \\ \|x\| = 1. \end{cases} \tag{1.2}$$

Here $\|\cdot\|$ is the Euclidean norm. If λ and x are complex, then they are called E-eiganvalue and E-eigenvector.

In the same year, Lim[94] independently defined eigenvalues for general real tensors in the real field. In his work, the l^k eigenvalues are H-eigenvalues, while the l^2 eigenvalues of tensors are the Z-eigenvalues. Note that in the case of $m = 2$, both the H-eigenvalue and Z-eigenvalue collapse to the eigenvalues of a matrix.

The concept of singular values and singular vectors can be generalized to higher order tensors as well. Let us recall in the real case that for a matrix $A \in \mathbb{R}^{n_1 \times n_2}$, if there exists a triple $(\lambda, x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ such that

$$Ax_2 = \lambda x_1 \text{ and } A^{\mathrm{T}} x_1 = \lambda x_2,$$

then λ is called a singular value of A, and x_1 and x_2 are the left and right singular vector corresponding to λ, respectively.

For higher order tensors, for example, consider a third order tensor $\mathcal{A}\in\mathbb{R}^{n_1\times n_2\times n_3}$. The corresponding singular value problem can be defined as[94]

$$\begin{cases} \mathcal{A}x_2x_3 = \lambda x_1, \\ \mathcal{A}x_1x_3 = \lambda x_2, \\ \mathcal{A}x_1x_2 = \lambda x_3, \\ \|x_1\| = 1, \|x_2\| = 1, \|x_3\| = 1, \lambda \in \mathbb{R}, x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}, x_3 \in \mathbb{R}^{n_3}, \end{cases}$$

where

$$(\mathcal{A}x_2x_3)_i = \sum_{j=1}^{n_2}\sum_{k=1}^{n_3} a_{ijk}(x_2)_j(x_3)_k,$$

$$(\mathcal{A}x_1x_3)_j = \sum_{i=1}^{n_1}\sum_{k=1}^{n_3} a_{ijk}(x_1)_i(x_3)_k,$$

$$(\mathcal{A}x_1x_2)_k = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} a_{ijk}(x_1)_i(x_2)_j.$$

For other higher order tensors, the definition is similar.

There are other kinds of eigenvalues/singular values for other types of tensors. Assume that p, q, n_1 and n_2 are positive integers, and $n_1, n_2 \geqslant 2$. We call $\mathcal{A} = (a_{i_1\cdots i_p j_1\cdots j_q})$, where $a_{i_1\cdots i_p j_1\cdots j_q} \in \mathbb{R}$, for $i_k = 1, \cdots, n_1$, $k = 1, \cdots, p$, and $j_k = 1, \cdots, n_2$, $k = 1, \cdots, q$, a real (p, q)-th order $n_1 \times n_2$ dimensional rectangular tensor, or simply a real rectangular tensor. When $p = q = 1$, $\mathcal{A}$ is simply a real $n_1 \times n_2$ rectangular matrix. Denote $M = p + q$. If there is a complex number λ and two nonzero complex vectors x and y such that:

$$\begin{cases} \mathcal{A}x^{p-1}y^q = \lambda x^{[M-1]}, \\ \mathcal{A}x^p y^{q-1} = \lambda y^{[M-1]}, \end{cases}$$

where $\mathcal{A}x^{p-1}y^q$ is a vector in $\mathbb{R}^{n_1}$ whose i-th entry is given by

$$(\mathcal{A}x^{p-1}y^q)_i = \sum_{i_2,\cdots,i_p=1}^{n_1} \sum_{j_1,\cdots,j_q=1}^{n_2} a_{ii_2\cdots i_p j_1\cdots j_q} x_{i_2}\cdots x_{i_p} y_{j_1}\cdots y_{j_q},$$

and $\mathcal{A}x^p y^{q-1}$ is a vector in $\mathbb{R}^{n_2}$ whose j-th entry is given by

$$(\mathcal{A}x^p y^{q-1})_j = \sum_{i_1,\cdots,i_p=1}^{n_1} \sum_{j_2,\cdots,j_q=1}^{n_2} a_{i_1\cdots i_p j j_2\cdots j_q} x_{i_1}\cdots x_{i_p} y_{j_2}\cdots y_{j_q},$$

then λ is called the singular value of $\mathcal{A}$ and (x, y) are the left and right eigenvectors of $\mathcal{A}$, associated with λ, respectively. If $(\lambda, x, y) \in \mathbb{R} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, then λ is an H-singular value of $\mathcal{A}$, and (x, y) are the left and right H-eigenvectors associated with λ. If a singular value is not an H-singular value, then it is an N-singular value of $\mathcal{A}$[27].

Assume that $\mathcal{A}$ is a 4-th order partially symmetric tensor, i.e., every entry of $\mathcal{A}$ satisfies $a_{ijkl} = a_{jikl} = a_{ijlk}$. The M-eigenvalue system of $\mathcal{A}$ is defined as

$$\begin{cases} \mathcal{A}xyy = \lambda x, \\ \mathcal{A}xxy = \lambda y, \\ \|x\| = 1, \|y\| = 1, x \in \mathbb{R}^{n_1}, y \in \mathbb{R}^{n_2}. \end{cases}$$

Here $\lambda \in \mathbb{R}$ is called the M-eigenvalue of $\mathcal{A}$. This type of eigenvalue was introduced by Qi et al.[122], which finds applications in nonlinear elastic materials analysis[122] and quantum entanglement problems in quantum physics[35].

There are other type of eigenvalues, e.g, the Z_1-eigenvalue of a transition probability tensor was introduced by Li and and Ng[90], where a transition probability tensor $\mathcal{P}$ of order m dimension n is defined to satisfy $\sum_{i_1=1}^{n} p_{i_1 i_2 \cdots i_m} = 1$, and the Z_1-eigenvalue problem is replacing the 2-norm in the Z-eigenvalue problem (1.2) by the 1-norm, i.e., the Z_1-eigenvalue problem is defined by

$$\begin{cases} \mathcal{A}x^{m-1} = \lambda x, \\ \|x\|_1 = 1. \end{cases} \tag{1.3}$$

This type of eigenvalue problems can be used to approximate the limit probability distribution vector of a higher order Markov chain[90].

1.2 Related polynomial optimization problems

Before proceeding, we first introduce the symmetric property of higher order tensors. A tensor $\mathcal{A}$ is called symmetric, if every of its entry $a_{i_1 \cdots i_m}$ is invariant under all permutations of $\{i_1, \cdots, i_m\}$, i.e., the following equalities always hold:

$$a_{i_1 \cdots i_m} = a_{\sigma(i_1 \cdots i_m)},$$

where $\sigma(\cdot)$ is the permutation function.

Every symmetric tensor is related to a polynomial function uniquely[33]. Specifically, suppose that $\mathcal{A}$ is an order m n-th dimensional symmetric tensor. Then

$$f_{\mathcal{A}}(x) = \mathcal{A}x^m = \sum_{i_1, \cdots, i_m = 1}^{n} a_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m}$$

is the associated polynomial function.

Consider the following homogenous polynomial optimization problem of degree m:

$$\begin{aligned} \max \quad & \mathcal{A}x^m, \\ \text{s.t.} \quad & \|x\| = 1, x \in \mathbb{R}^n. \end{aligned} \tag{1.4}$$

The corresponding Lagrangian function of (1.4) is given by

$$L(x,\lambda) = \mathcal{A}x^m - \lambda(x^{\mathrm{T}}x - 1),$$

where $\lambda \in \mathbb{R}$ is the Lagrangian multiplier. Taking the gradient of $L(\cdot,\cdot)$ with respect to x and λ respectively and setting them both to zero, we get the following KKT system of (1.4)

$$\begin{cases} \mathcal{A}x^{m-1} = \dfrac{2\lambda}{m}x, \\ x^{\mathrm{T}}x = 1, \end{cases}$$

which is in fact the Z-eigenvalue problem of $\mathcal{A}$ if we let $\lambda := 2\lambda/m$ in the above system. Thus finding the largest Z-eigenvalue of $\mathcal{A}$ is equivalent to solving the polynomial optimization problem (1.4).

Suppose the symmetric tensor $\mathcal{A}$ is of even order. order Then the KKT system of the following problem

$$\begin{aligned} \max \quad & \mathcal{A}x^m, \\ \text{s.t.} \quad & \|x\|_m = 1, x \in \mathbb{R}^n \end{aligned}$$

is

$$\begin{cases} \mathcal{A}x^{m-1} = \lambda x^{[m-1]}, \\ \sum_{i=1}^n x_i^m = 1, \end{cases}$$

where $\|x\|_m = (\sum_{i=1}^n |x_i|^m)^{\frac{1}{m}}$. Note that the first formula is homogenous with respect to x. Thus the second formula can be eliminated, which is the H-eigenvalue problem of $\mathcal{A}$.

Finding the largest M-eigenvalue of a 4-th order partially symmetric tensor $\mathcal{A}$ is equivalent to solving the following homogeneous polynomial problem with degree 4:

$$\begin{aligned} \max \quad & \mathcal{A}xxyy = \sum_{i,j=1}^{n_1} \sum_{k,l=1}^{n_2} a_{ijkl}x_ix_jy_ky_l, \\ \text{s.t.} \quad & \|x\| = 1, \|y\| = 1, x \in \mathbb{R}^{n_1}, y \in \mathbb{R}^{n_2}. \end{aligned} \tag{1.5}$$

This is known as the biquadratic optimization problem[62, 97, 122].

Similarly, finding the largest singular value of a tensor of order d is equivalent to solving the following multi-linear optimization problem:

$$\begin{aligned} \max \quad & \mathcal{A}x_1 \cdots x_d = \sum_{i_1=1}^{n_1} \cdots \sum_{i_d=1}^{n_d} a_{i_1\cdots i_d}(x_1)_{i_1} \cdots (x_d)_{i_d}, \\ \text{s.t.} \quad & \|x^1\| = 1, \cdots, \|x^d\| = 1, x^i \in \mathbb{R}^{n_i}, i = 1, \cdots, d. \end{aligned} \tag{1.6}$$

The polynomial optimization in binary variables or in mixed variables was studied in [54,56]:

$$\begin{aligned} \max \quad & \mathcal{A}x^m, \\ \text{s.t.} \quad & x \in \{-1,1\}^n, \end{aligned} \tag{1.7}$$

$$\begin{aligned} \max \quad & \mathcal{A}x_1 \cdots x_d, \\ \text{s.t.} \quad & x_i \in \{-1,1\}^{n_i}, i = 1, \cdots, d, \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} \max \quad & \mathcal{A}x_1 \cdots x_d, \\ \text{s.t.} \quad & \|x_i\| = 1, x_i \in \mathbb{R}^{n_i}, i = 1, \cdots, l, \\ & x_j \in \{-1,1\}^{n_j}, j = l+1, \cdots, d, \end{aligned} \tag{1.9}$$

which relate to the $\infty \to 1$-norm of higher order tensors[56].

The biquadratic optimization over quadratic constraints or standard simplex was studied by Zhang et al.[169], Ling et al.[98] and Bomze et al.[7]:

$$\begin{aligned} \max \quad & \mathcal{A}xxyy, \\ \text{s.t.} \quad & x^{\mathrm{T}} A^i x \leqslant 1, \quad i = 1, \cdots, p, \\ & y^{\mathrm{T}} B^j y \leqslant 1, \quad j = 1, \cdots, q, \end{aligned} \tag{1.10}$$

$$\begin{aligned} \max \quad & \mathcal{A}xxyy, \\ \text{s.t.} \quad & \sum_{i=1}^m x_i = 1, x \geqslant 0, \\ & \sum_{j=1}^n y_j = 1, y \geqslant 0, \end{aligned} \tag{1.11}$$

which arise from portfolio selection[98].

1.3 Applications

This section lists some applications of eigenvalues and singular values of tensors and their related polynomial optimization problems.

Application of the H-eigenvalue to maximum clique problem[14,15]:

A k-uniform hypergraph, or simply a k-graph, is a pair $G = (V, E)$, where $V = \{1, \cdots, n\}$ is a finite set of vertices and $E \subseteq \binom{V}{k}$ is a set of k-subsets of V, each of which is called a hyperedge. Particularly, a 2-graph is the usual graph. A subset of vertices $C \subseteq V$ is called a hyperclique if $\binom{C}{k} \subseteq E$. A clique is said to be maximal if it is not contained in any other clique, while it is called maximum if it has maximum cardinality. Specifically, the maximal clique problem of a 2-graph is to find the largest subset of vertices in which each point is directly connected to every other vertex in the subset. The clique number of a graph G, denoted by $\omega(G)$, is defined as the cardinality of a maximum clique. Computing $\omega(G)$ is NP-hard. Bulò and Pelillo[15] provided a new upper bound for $\omega(G)$ of an undirected graph G:

$$\omega(G) \leqslant \frac{\rho(\mathcal{A})}{k!} + k,$$

where $\mathcal{H}$ is the induced tensor of a k-clique $(k+1)$-graph constructed from G, and $\rho(\mathcal{H})$ is the largest H-eigenvalue of $\mathcal{H}$. A new lower bound was also provided via the largest H-eigenvalue.

Application of singular value and Z-eigenvalue to the best rank-one approximation of higher order tensors[36, 78].

In signal processing, data analysis, chemometrics, and psychology, data may sometimes be organized as higher order higher dimension tensors. How to extract useful information from this data is an important problem[36]. One possible way is to find a rank-one tensor which is the closest one to the original tensor under the Euclidean distance, which contains most of the information of the original tensor. To be more specific, let the original d-th order tensor be $\mathcal{F}$ and the rank-one tensor be $\mathcal{B} = x^1 \otimes x^2 \otimes \cdots \otimes x^d$, where "$\otimes$" denotes the Kronecker product, and $b_{i_1 \cdots i_d} = x^1_{i_1} \cdots x^d_{i_d}$. The associated problem can be written as:

$$\begin{aligned} \min \quad & \|\mathcal{F} - \mathcal{B}\|_F = \|\mathcal{F} - x_1 \otimes \cdots \otimes x_d\|_F, \\ \text{s.t.} \quad & x_i \in \mathbb{R}^{n_i}, i = 1, \cdots, d, \end{aligned}$$

where $\|\mathcal{F}\|_F = \sqrt{\sum_{i_1=1}^{n_1} \cdots \sum_{i_d=1}^{n_d} f^2_{i_1 \cdots i_d}}$. If we restrict the norm of x^i to be one, then it can be rewritten as

$$\begin{aligned} \min \quad & \|\mathcal{F} - \lambda x_1 \otimes \cdots \otimes x_d\|_F, \\ \text{s.t.} \quad & \lambda \in \mathbb{R}, \|x_i\| = 1, x_i \in \mathbb{R}^{n_i}, i = 1, \cdots, d. \end{aligned}$$

Note that

$$\begin{aligned} & \|\mathcal{F} - \lambda x_1 \otimes \cdots \otimes x_d\|_F^2 \\ = & \|\mathcal{F}\|_F^2 - 2\lambda \cdot \mathcal{F} x_1 \cdots x_d + \lambda^2 \|x_1 \otimes \cdots \otimes x_d\|_F^2 \\ = & \lambda^2 - 2\lambda \cdot \mathcal{F} x_1 \cdots x_d + \|\mathcal{F}\|_F^2. \end{aligned}$$

Minimizing this with respect to λ, we get

$$\|\mathcal{F} - \lambda x_1 \otimes \cdots \otimes x_d\|_F^2 = \|\mathcal{F}\|_F^2 - (\mathcal{F} x_1 \cdots x_d)^2,$$

which reduces to

$$\begin{aligned} \max \quad & |\mathcal{F}x_1 \cdots x_d| = \mathcal{F}x_1 \cdots x_d, \\ \text{s.t.} \quad & \|x_1\| = 1, \cdots, \|x_d\| = 1, x^i \in \mathbb{R}^n, i = 1, \cdots, d, \end{aligned}$$

which reduces to finding the largest singular value of $\mathcal{F}$.

Suppose the symmetric tensor $\mathcal{A}$ is of order m dimension n. The best symmetric rank-one approximation to $\mathcal{A}$ is

$$\begin{aligned} \min \quad & \|\mathcal{A} - \lambda x \otimes \cdots \otimes x\|_F, \\ \text{s.t.} \quad & \lambda \in \mathbb{R}, \|x\| = 1, x \in \mathbb{R}^n. \end{aligned}$$

Using a similar argument, the problem is equivalent to

$$\begin{aligned} \max \quad & |\mathcal{A}x^m|, \\ \text{s.t.} \quad & \|x\| = 1, \end{aligned}$$

which is finding the extreme Z-eigenvalue of $\mathcal{A}$.

In automatic control, one needs to determine whether a quartic form is positive semidefinite or not[110]. Specifically, for any $x \in \mathbb{R}^n$, to determine whether

$$\sum_{i,j,k,l=1}^{n} a_{ijkl} x_i x_j x_k x_l \geqslant 0$$

or not. by restricting x on the unit sphere, it can be written as the following problem:

$$\begin{aligned} \min \quad & \mathcal{A}x^4 := \sum_{i,j,k,l=1}^{n} a_{ijkl} x_i x_j x_k x_l, \\ \text{s.t.} \quad & \|x\| = 1,\ x \in \mathbb{R}^n, \end{aligned}$$

which is finding the smallest Z-eigenvalue of $\mathcal{A}$.

The M-eigenvalue finds applications in nonlinear elastic materials analysis and quantum physics. In nonlinear elastic materials where one needs to determines whether the strong or ordinary ellipticity condition holds or not, where the strong one holds if and only if the minimum of a biquadratic function over unit spheres in $\mathbb{R}^3$ is positive:

$$\min \left\{ f(x, y) = \sum_{i,j,k,l=1}^{3} a_{ijkl} x_i x_j y_k y_l \;\middle|\; \|x\| = 1, \|y\| = 1 \right\} > 0,$$

and the ordinary one holds if and only if the minimum is nonnegative. In quantum physics, where a quantum state is separable or entangled is a fundamental and