Graduate Texts in Mathematics

M.Scott Osborne

Basic Homological Algebra

基本同调代数

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Preface

Five years ago, I taught a one-quarter course in homological algebra. I discovered that there was no book which was really suitable as a text for such a short course, so I decided to write one. The point was to cover both Ext and Tor early, and still have enough material for a larger course (one semester or two quarters) going off in any of several possible directions. This book is also intended to be readable enough for independent study.

The core of the subject is covered in Chapters 1 through 3 and the first two sections of Chapter 4. At that point there are several options. Chapters 4 and 5 cover the more traditional aspects of dimension and ring changes. Chapters 6 and 7 cover derived functors in general. Chapter 8 focuses on a special property of Tor. These three groupings are independent, as are various sections from Chapter 9, which is intended as a source of special topics. (The prerequisites for each section of Chapter 9 are stated at the beginning.)

Some things have been included simply because they are hard to find elsewhere, and they naturally fit into the discussion. Lazard's theorem (Section 8.4) is an example; Sections 4, 5, and 7 of Chapter 9 contain other examples, as do the appendices at the end.

The idea of the book's plan is that subjects can be selected based on the needs of the class. When I taught the course, it was a prerequisite for a course on noncommutative algebraic geometry. It was also taken by several students interested in algebraic topology, who requested the material in Sections 9.2 and 9.3. (One student later said he wished he'd seen injective envelopes, so I put them in, too.) The ordering of the subjects in Chapter

9 is primarily based on how involved each section's prerequisites are.

The prerequisite for this book is a graduate algebra course. Those who have seen categories and functors can skip Chapter 1 (after a peek at its appendix).

There are a few oddities. The chapter on abstract homological algebra, for example, follows the pedagogical rule that if you don't need it, don't define it. For the expert, the absence of pullbacks and pushouts will stand out, but they are not needed for abstract homological algebra, not even for the long exact sequences in Abelian categories. In fact, they obscure the fact that, for example, the connecting morphism in the ker-coker exact sequence (sometimes called the snake lemma) is really a homology morphism. Similarly, overindulgence in δ-functor concepts may lead one to believe that the subject of Section 6.5 is moot.

In the other direction, more attention is paid (where necessary) to set theoretic technicalities than is usual. This subject (like category theory) has become widely available of late, thanks to the very readable texts of Devlin [15], Just and Weese [41], and Vaught [73]. Such details are not needed very often, however, and the discussion starts at a much lower level.

Solution outlines are included for some exercises, including exercises that are used in the text.

In preparing this book, I acknowledge a huge debt to Mark Johnson. He read the whole thing and supplied numerous suggestions, both mathematical and stylistic. I also received helpful suggestions from Garth Warner and Paul Smith, as well as from Dave Frazzini, David Hubbard, Izuru Mori, Lee Nave, Julie Nuzman, Amy Rossi, Jim Mailhot, Eric Rimbey, and H. A. R. V. Wijesundera. Kate Senehi and Lois Fisher also supplied helpful information at strategic points. Many thanks to them all. I finally wish to thank Mary Sheetz, who put the manuscript together better than I would have believed possible.

Concerning source material, the very readable texts of Jans [40] and Rotman [68] showed me what good exposition can do for this subject, and I used them heavily in preparing the original course. I only wish I could write as well as they do.

M. Scott Osborne
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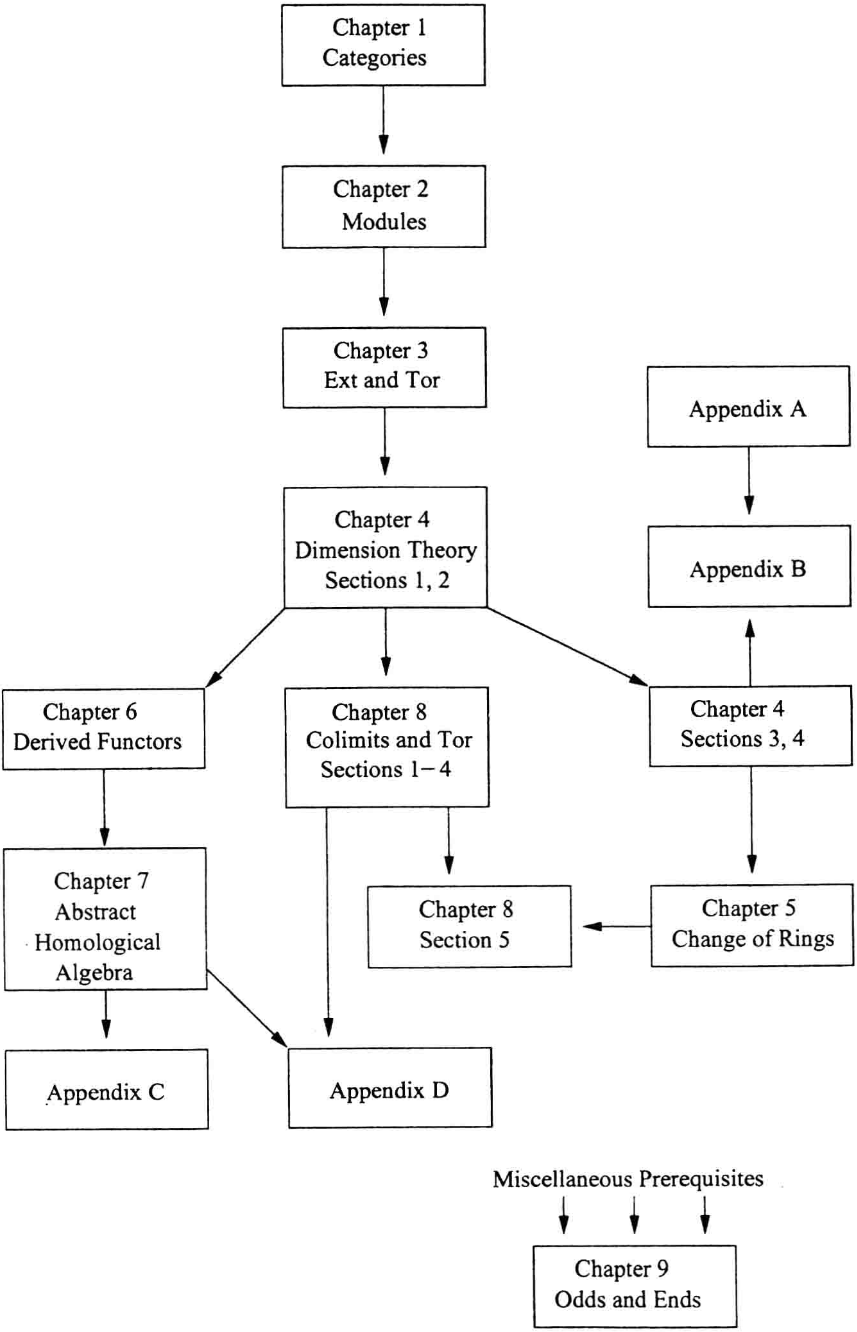
Fall, 1998

Chapter/Appendix Dependencies



Contents

1
Categories

Homological algebra addresses questions that appear naturally in category theory, so category theory is a good starting point. Most of what follows is standard, but there are a few slippery points.

First, a few words about classes. The concept of a *class* is intended to generalize the concept of a set. That is, not only will all sets be classes, but some other collections of things that are "too big" to be sets will also be classes. For example, the collection of all sets is a class. It is a *proper* class, in the sense that it cannot be a set; this is the Russell paradox, which traditionally is presented as follows.

Let **S** be the class of all sets. Assume **S** is a set. Then

$$A = \{X \in \mathbf{S} \mid X \notin X\}$$

is also a set. Note that for any set X,

$$X \in A \Leftrightarrow X \notin X.$$

In particular, taking $X = A$,

$$A \in A \Leftrightarrow A \notin A,$$

a contradiction.

Note also that $\mathcal{P}(\mathbf{S}) \subset \mathbf{S}$, which should be bizarre enough.

In Gödel–Bernays–von Neumann class theory, sets are defined as classes which are members of other classes. In fact, the *only* members any class has are sets. The power class is the collection of sub*sets*, so $\mathcal{P}(\mathbf{S}) = \mathbf{S}$, and $\mathbf{S} \notin \mathcal{P}(\mathbf{S})$. The axioms of Gödel–Bernays–von Neumann class theory lead to what we have learned to expect of classes, but this is a complicated business. A brief variant appears as an appendix to Kelley [48, pp. 250–281].

For our purposes (at least until Section 6.6), all we need to know is that the class concept is like the set concept, only broader: Classes are still collections of things, and all sets are classes, but some classes (like the class of all sets) are not sets. Also, the *elementary* set manipulations, like union, intersection, specification, formation of functions, etc., can be carried out for classes as well. The one thing we *cannot* do is force a class to belong to another class, unless the first class is actually a set. For example, one can define an equivalence relation on a class, and then form equivalence classes, but one cannot form the class of equivalence classes unless the equivalence classes are actually sets. An example on the class $\mathbf{S}$: Say that $X \sim Y$ when X and Y have the same cardinality. The equivalence class of $\emptyset$ is $\{\emptyset\}$; it is the only equivalence class which is a set.

A *category* $\mathbf{C}$ consists of a class of objects, $\operatorname{obj}\mathbf{C}$, together with sets (repeat, *sets*) of morphisms, which arise in the following manner. There is a function Mor which assigns to each pair $A, B \in \operatorname{obj}\mathbf{C}$ a set of morphisms $\operatorname{Mor}(A,B)$ from A to B, sometimes written $\operatorname{Mor}_{\mathbf{C}}(A,B)$ if $\mathbf{C}$ is to be emphasized. $\operatorname{Mor}(A,B)$ is called the set of *morphisms from A to B*. The category $\mathbf{C}$ also includes a pairing (function), called composition:

$$\operatorname{Mor}(B,C) \times \operatorname{Mor}(A,B) \to \operatorname{Mor}(A,C)$$

$$(g,f) \mapsto gf.$$

Finally, each $\operatorname{Mor}(A,A)$ contains a distinguished element i_A. The axioms are:

1) Composition is *associative*. That is, if $f \in \operatorname{Mor}(C,D)$, $g \in \operatorname{Mor}(B,C)$, and $h \in \operatorname{Mor}(A,B)$, then $(fg)h = f(gh)$.

2) Each i_A is an *identity*. That is, if $f \in \operatorname{Mor}(A,B)$, then $f = fi_A = i_B f$.

Note: Many authors also require

3) $\operatorname{Mor}(A,B)$ is disjoint from $\operatorname{Mor}(C,D)$ unless $A = C$, $B = D$.

This serves as a bookkeeping device, and also allows certain constructions. It is also a pain in the neck to enforce. (See below concerning concrete categories.) However, if $\mathbf{C}$ does not satisfy this, one may replace $f \in \operatorname{Mor}(A,B)$ by the ordered triple (A,f,B). That is, replace $\operatorname{Mor}(A,B)$ by $\{A\} \times \operatorname{Mor}(A,B) \times \{B\}$.

Example 1 SETS. $\operatorname{obj}\mathbf{Set}$ = class of all sets. $\operatorname{Mor}(A,B)$ = all functions from A to B.

Example 2 GROUPS. obj **Gr** = class of all groups. $\mathrm{Mor}(A, B)$ = all homomorphisms from A to B.

Example 3 TOPOLOGICAL SPACES. obj **Top** = class of all topological spaces. $\mathrm{Mor}(A, B)$ = all continuous $f : A \to B$.

"Composition" is functional composition. The reader should be able to provide lots of examples like the above. There are other kinds, as well.

Example 4 Given $\mathbf{C}$, form the opposite category, $\mathbf{C}^{\mathrm{op}}$: obj $\mathbf{C}$ = obj $\mathbf{C}^{\mathrm{op}}$, while $\mathrm{Mor}_{\mathbf{C}^{\mathrm{op}}}(A, B) = \mathrm{Mor}_{\mathbf{C}}(B, A)$. Composition is reversed: Letting $*$ denote composition in $\mathbf{C}^{\mathrm{op}}$, set $f * g = gf$.

Example 5 Note that from the definition, $\mathrm{Mor}(A, A)$ is always a monoid, that is, a semigroup with identity. This is quite general; if S is a monoid, define a category as follows: obj $\mathbf{C} = \{S\}$, and set $\mathrm{Mor}(S, S) = S$. Composition is the semigroup multiplication. Note further that the singleton obj $\mathbf{C}$ can, in fact, be replaced by any other singleton $\{A\}$, with $\mathrm{Mor}(A, A) = S$. At this point, we are rather far from our intuitions about morphisms; the circuit breakers in our heads may need resetting.

The last two examples are different in flavor from the first three. But that's good; the notion of a category is broad enough to include some weird examples. To isolate the content of the first three examples:

Definition 1.1 *A category* $\mathbf{C}$ *is called a* **concrete** *category provided* $\mathbf{C}$ *comes equipped with a function* σ *whose domain is* obj $\mathbf{C}$ *such that*

1. *If* $A \in$ obj $\mathbf{C}$, *then* $\sigma(A)$ *is a set. (It is called the* underlying set *of* A.*)*

2. $\mathrm{Mor}(A, B)$ *consists of functions from* $\sigma(A)$ *to* $\sigma(B)$, *that is, any* $f \in \mathrm{Mor}(A, B)$ *is a function from* $\sigma(A)$ *to* $\sigma(B)$.

3. *Categorical composition is functional composition.*

4. i_A *is the identity map on* $\sigma(A)$.

Observe that, if one adopts the disjointness requirement in the definition of a category, condition 2 cannot be taken literally. (For example, in **Set**, $\mathrm{Mor}(\emptyset, A) = \{\text{empty function}\}$ for all A.) Rather, replace it with

2′. $\mathrm{Mor}(A, B)$ *consists of ordered triples* (A, f, B), *where* f *is a function from* $\sigma(A)$ *to* $\sigma(B)$.

Concrete categories have a number of uses; an odd one will be described in the appendix. One use is the definition of free objects.

Definition 1.2 *If* $\mathbf{C}$ *is a concrete category,* $F \in \operatorname{obj}\mathbf{C}$*, and* X *is a set, and if* $\varphi : X \to \sigma(F)$ *is a one-to-one function, then* F *is called* **free** *on* X *if and only if for every* $A \in \operatorname{obj}\mathbf{C}$ *and set map* $\psi : X \to \sigma(A)$*, there exists a unique morphism* $f \in \operatorname{Mor}(F, A)$ *such that* $f \circ \varphi = \psi$ *as set maps from* X *to* $\sigma(A)$.

Example 6 $\mathbf{C} = \mathbf{Ab} =$ category of Abelian groups. Say $X = \{1,2\}$, $\sigma(F) = \mathbb{Z} \times \mathbb{Z}$, i.e., F is the Abelian group $\mathbb{Z} \times \mathbb{Z}$. Define $\varphi(1) = (1,0)$ and $\varphi(2) = (0,1)$. Given $\psi : X \to \sigma(A)$, with $\psi(1) = a$, $\psi(2) = b$, define $f : F \to A$ by $f(m,n) = ma + nb$. Then $f \circ \varphi = \psi$; furthermore, the definition of f is forced. Roughly speaking, F is "large enough" so that f *can* be defined, while F is not "too large" so that f is unique.

One quick definition: If $\mathbf{C}$ is a category,, and $f \in \operatorname{Mor}(A, B)$, then f is an *isomorphism* provided there is a $g \in \operatorname{Mor}(B, A)$ for which $fg = i_B$ and $gf = i_A$. By the usual trickery, g is unique: Given just that $fg' = i_B$, then $g' = i_A g' = (gf)g' = g(fg') = g i_B = g$.

Theorem 1.3 *If* X *is a set,* $\mathbf{C}$ *is a concrete category, and* F, F' *are free on* X *(with* $\varphi : X \to \sigma(F)$, $\varphi' : X \to \sigma(F')$*), then* F *and* F' *are isomorphic.*

Proof: F being free, $\exists\ f \in \operatorname{Mor}(F, F')$ with $\varphi' = f \circ \varphi$. F' being free, $\exists\ g \in \operatorname{Mor}(F', F)$ with $\varphi = g \circ \varphi'$. Then $gf \in \operatorname{Mor}(F, F)$. Also, $\varphi = g\varphi' = g(f\varphi) = (gf)\varphi$. The uniqueness of the map (namely i_F) satisfying $\varphi = h\varphi$ implies that $gf = i_F$. Similarly, $fg = i_{F'}$. □

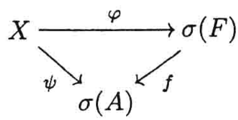
The above can be illustrated by using diagrams, as will usually be done in what follows. A $\varphi \in \operatorname{Mor}(A, B)$ can be illustrated by an arrow:

$$A \xrightarrow{\ \varphi\ } B \qquad \text{or} \qquad \begin{array}{c} A \\ \varphi \big\downarrow \\ B \end{array} \qquad \text{or} \qquad \begin{array}{ccc} B & & \\ & \nwarrow^{\varphi} & \\ & & A \end{array} \qquad \cdots$$

Diagrams assemble such morphisms:

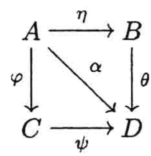
$$\begin{array}{ccc} A & \xrightarrow{\ \ \varphi\ \ } & C \\ & {}_{\psi}\searrow \quad \nearrow_{\theta} & \\ & B & \end{array}$$

A diagram is *commutative* if any two paths along arrows that start at the same point and finish at the same point yield the same morphism via composition along successive arrows. In the diagram above, two paths lead from A to C, the direct one and the indirect one, so commutativity requires $\varphi = \theta\psi$. For example, the commutative diagram associated with the definition of a free object is



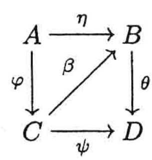
which illustrates the concept more clearly than the prose in the definition.

There may be many paths:



Commutativity requires $\alpha = \psi\varphi = \theta\eta$.

There may be many initial and/or final points:



Commutativity requires $\eta = \beta\varphi$ and $\psi = \theta\beta$, as well as $\theta\eta = \psi\varphi$. The last follows from the first two: $\theta\eta = \theta\beta\varphi = \psi\varphi$. That is, commutativity of the whole diagram follows from commutativity of the two triangles. This phenomenon is common; complicated diagrams are checked for commutativity by checking indecomposable pieces.

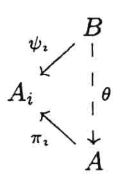
Diagrams are so useful that it may (depending on psychological factors more than anything else) be helpful to visualize morphisms as literal arrows.

Suppose $\{A_i : i \in \mathcal{I}\}$ is an indexed family from obj $\mathbf{C}$. A *product* of the A_i, written

$$\prod_{i \in \mathcal{I}} A_i$$

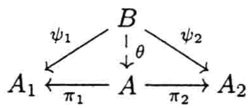
is an object A, together with morphisms $\pi_i \in \mathrm{Mor}(A, A_i)$ for all $i \in \mathcal{I}$, satisfying the following universal property:

If $B \in \mathrm{obj}\,\mathbf{C}$, and $\psi_i \in \mathrm{Mor}(B, A_i)$ for all $i \in \mathcal{I}$, then there is a unique $\theta \in \mathrm{Mor}(B, A)$ making all the diagrams

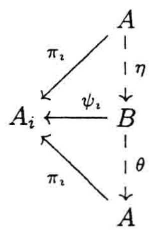


commutative. (A dashed line is used for θ to emphasize that its existence is being hypothesized. Such hypothesized morphisms are often called *fillers*. The idea is that a filler extends a diagram in a commutative way.) Roughly speaking, a single morphism into $\prod A_i$ models a collection of morphisms into the individual A_i ($\langle\psi_i\rangle \leftrightarrow \theta$); as a target of morphisms, $\prod A_i$ encapsulates all the A_i. In **Set**, the ordinary Cartesian product *is* a category theoretic product, with $\theta(b) = \langle\psi_i(b)\rangle$.

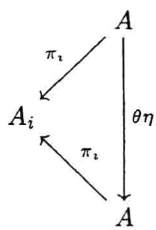
The case of only two objects can be illustrated with a single diagram:



If the indefinite article in the definition of a product worries you, and it should, rest assured. While products are not totally unique, they are as unique as could be hoped for—they are unique up to isomorphism. For if the B above is also a product, then there is also a unique η making



commutative, whence



is commutative. Uniqueness implies that $\theta\eta = i_A$. Similarly $\eta\theta = i_B$.

The above is an example of a universal mapping construction; in general, there are morphisms between some of the A_i. There may even be noncategorical things, like the bilinear maps used to define tensor products. The idea behind uniqueness is the same, however, and such objects are unique up to isomorphism when the recipe allows the above argument to work. Here it is more important to understand the principle than to have a general theorem stated.

Coproducts are just products on the opposite category. The coproduct of A_i is an A, together with $\varphi_i \in \mathrm{Mor}(A_i, A)$. The diagrams that must