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Calculus on Normed Vector Spaces

赋范向量空间上的微积分

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Calculus on Normed Vector Spaces
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著　　者： Rodney Coleman
责任编辑： 刘　慧　岳利青
装帧设计： 任志远

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Rodney Coleman
Laboratoire Jean Kuntzmann
Grenoble
France

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To Francie, Guillaume, Samuel and Jeremy

Preface

The aim of this book is to present an introduction to calculus on normed vector spaces at a higher undergraduate or beginning graduate level. The prerequisites are basic calculus and linear algebra. However, a certain mathematical maturity is also desirable. All the necessary topology and functional analysis is introduced where necessary.

I have tried to show how calculus on normed vector spaces extends the basic calculus of functions of several variables. I feel that this is often not done and we have, on the one hand, very elementary texts, and on the other, high level texts, but few bridging the gap.

In the text there are many nontrivial applications of the theory. Also, I have endeavoured to give exercises which seem, at least to me, interesting. In my experience, very often the exercises in books are trivial or very academic and it is difficult to see where the interest lies (if there is any!).

In writing this text I have been influenced and helped by many other works on the subject and by others close to it. In fact, there are too many to mention; however, I would like to acknowledge my debt to the authors of these works. I would also like to express my thanks to Mohamed El Methni and Sylvain Meignen, who carefully read the text and gave me many helpful suggestions.

Writing this book has allowed me to clarify many of my ideas and it is my sincere hope that this work will prove useful in aiding others.

Grenoble, France

Rodney Coleman

Contents

Chapter 1
Normed Vector Spaces

In this chapter we will introduce normed vector spaces and study some of their elementary properties.

1.1 First Notions

We will suppose that all vector spaces are real. Let E be a vector space. A mapping $\|\cdot\| : E \longrightarrow \mathbb{R}$ is said to be a *norm* if, for all $x, y \in E$ and $\lambda \in \mathbb{R}$ we have

- $\|x\| \geq 0$;
- $\|x\| = 0 \Longleftrightarrow x = 0$;
- $\|\lambda x\| = |\lambda|\|x\|$;
- $\|x + y\| \leq \|x\| + \|y\|$.

The pair $(E, \|\cdot\|)$ is called a *normed vector space* and we say that $\|x\|$ is the norm of x. The fourth property is often referred to as the normed vector space *triangle inequality*.

Exercise 1.1. Show that $\|-x\| = \|x\|$ and that $\|\cdot\|$ is a convex function, i.e.,

$$\|\lambda x + (1 - \lambda)y\| \leq \lambda\|x\| + (1 - \lambda)\|y\|$$

for all $x, y \in E$ and $\lambda \in [0, 1]$.

When there is no confusion, we will simply write E for a normed vector space. To distinguish norms on different normed vector spaces we will often use suffixes. For example, if we are dealing with the normed vector spaces E and F, we may write $\|\cdot\|_E$ for the norm on E and $\|\cdot\|_F$ for the norm on F. The most common norms on $\mathbb{R}^n$ are defined as follows:

$$\|x\|_1 = |x_1| + \cdots + |x_n|, \quad \|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2} \quad \text{and}$$
$$\|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\},$$

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where x_i is the ith coordinate of x. There is no difficulty in seeing that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms. For $\|\cdot\|_2$ the only difficulty can be found in the last property. If we set $x \cdot y = \sum_{i=1}^n x_i y_i$, the dot product of x and y, and write $\|\cdot\|$ for $\|\cdot\|_2$, then

$$p(t) = t^2\|x\|^2 + 2t(x \cdot y) + \|y\|^2 = \|tx + y\|^2 \geq 0.$$

As p is a second degree polynomial and always nonnegative, we have

$$4\big((x \cdot y)^2 - \|x\|^2\|y\|^2\big) \leq 0$$

and so

$$\begin{aligned}\|x + y\|^2 &= (x + y) \cdot (x + y) = \|x\|^2 + \|y\|^2 + 2(x \cdot y)\\ &\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2.\end{aligned}$$

This gives us the desired inequality.

If $n = 1$, then these three norms are the same, i.e.,

$$\|x\|_1 = \|x\|_2 = \|x\|_\infty = |x|.$$

Exercise 1.2. Characterize the norms defined on $\mathbb{R}$.

In general, we will suppose that the norm on $\mathbb{R}$ is the absolute value.

It is possible to generalize the norms on $\mathbb{R}^n$ defined above. We suppose that $p > 1$ and for $x \in \mathbb{R}^n$ we set $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$.

Proposition 1.1. *$\|x\|_p$ is a norm on $\mathbb{R}^n$.*

Proof. It is clear that the first three properties of a norm are satisfied. To prove the triangle inequality, we proceed by steps. We first set $q = \frac{p}{p-1}$ and prove the following formula for strictly positive numbers a and b:

$$a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}.$$

For $k \in (0,1)$ let the function $f_k : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f_k(t) = k(t-1) - t^k + 1$. Then $f_k(1) = 0$ and $\frac{\mathrm{d}f_k}{\mathrm{d}t}(t) = k(t - t^{k-1})$. It follows that $f_k(t) \geq 0$ if $t \geq 1$ and so, for $k \in (0,1)$ and $t \geq 1$,

$$t^k \leq tk + (1 - k).$$

If we set $t = \frac{a}{b}$ and $k = \frac{1}{p}$ when $a \geq b$, or $t = \frac{b}{a}$ and $k = \frac{1}{q}$ when $a < b$, then we obtain the result.

The next step is to take $x, y \in \mathbb{R}^n \setminus \{0\}$ and set $a = (\frac{|x_i|}{\|x\|_p})^p$ and $b = (\frac{|y_i|}{\|y\|_q})^q$ in the formula. We obtain

$$\frac{|x_i y_i|}{\|x\|_p \|y\|_q} \leq \frac{1}{p}\left(\frac{|x_i|}{\|x\|_p}\right)^p + \frac{1}{q}\left(\frac{|y_i|}{\|y\|_q}\right)^q,$$

and then, after summing over i,

$$\sum_{i=1}^{n} |x_i||y_i| \leq \|x\|_p \|y\|_q.$$

This inequality clearly also holds when $x = 0$ or $y = 0$.

Now,

$$\|x + y\|_p^p \leq \sum_{i=1}^{n} |x_i||x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i||x_i + y_i|^{p-1}$$

and, using the inequality which we have just proved,

$$\begin{aligned}\sum_{i=1}^{n} |x_i||x_i + y_i|^{p-1} &\leq \|x\|_p \left(\sum_{i=1}^{n} |x_i + y_i|^{(p-1)q}\right)^{\frac{1}{q}} \\ &= \|x\|_p \left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{q}} \\ &= \|x\|_p \|x + y\|_p^{\frac{p}{q}}.\end{aligned}$$

In the same way

$$\sum_{i=1}^{n} |y_i||x_i + y_i|^{p-1} \leq \|y\|_p \|x + y\|_p^{\frac{p}{q}}$$

and so

$$\|x + y\|_p^p \leq (\|x\|_p + \|y\|_p)\|x + y\|_p^{\frac{p}{q}},$$

from which we obtain the triangle inequality. □

Exercise 1.3. Show that

$$\lim_{p\to 1} \|x\|_p = \|x\|_1 \qquad \text{and} \qquad \lim_{p\to\infty} \|x\|_p = \|x\|_\infty$$

for any $x \in \mathbb{R}^n$.

Let E be a vector space and $\mathcal{N}(E)$ the collection of norms defined on E. We define a relation $\sim$ on $\mathcal{N}(E)$ by writing $\|\cdot\| \sim \|\cdot\|^\times$ if there exist constants $\alpha > 0$ and $\beta > 0$ such that

$$\alpha\|x\|^\times \leq \|x\| \leq \beta\|x\|^\times$$

for all $x \in E$. This relation is an equivalence relation. If $\|\cdot\| \sim \|\cdot\|^{\times}$, then we say that the two norms are *equivalent*.

Exercise 1.4. Establish the inequalities $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq n\|x\|_\infty$ and deduce that these three norms on $\mathbb{R}^n$ are equivalent.

If S is a nonempty set, then a real-valued function d defined on the Cartesian product S^2 is said to be a *metric* (or *distance*) if it satisfies the following properties for all $x, y, z \in S^2$:

- $d(x, y) \geq 0$;
- $d(x, y) = 0 \iff x = y$;
- $d(x, y) = d(y, x)$;
- $d(x, y) \leq d(x, z) + d(z, y)$.

We say that the pair (S, d) is a *metric space* and that $d(x, y)$ is the *distance* from x to y. The fourth property is referred to as the (metric space) triangle inequality. If $(E, \|\cdot\|)$ is a normed vector space, then it easy to see that, if we set $d(x, y) = \|x - y\|$, then d defines a metric on E. Many of the ideas in this chapter can be easily generalized to general metric spaces.

Exercise 1.5. What is the distance from $A = (1, 1)$ to $B = (4, 5)$ for the norms we have defined on $\mathbb{R}^2$?

Consider a point a belonging to the normed vector space E. If $r > 0$, then the set

$$B(a, r) = \{x \in E : \|a - x\| < r\}$$

is called the *open ball* of centre a and radius r. For $r \geq 0$ the set

$$\bar{B}(a, r) = \{x \in E : \|a - x\| \leq r\}$$

is called the *closed ball* of centre a and radius r. In $\mathbb{R}$ the ball $B(a, r)$ (resp. $\bar{B}(a, r)$) is the open (resp. closed) interval of length $2r$ and centre a. We usually refer to balls in the plane $\mathbb{R}^2$ as *discs*.

Exercise 1.6. What is the form of the ball $\bar{B}(0, 1) \subset \mathbb{R}^2$ for the norms $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$? (Notice that a ball may have corners.)

1.2 Limits and Continuity

We now consider sequences in normed vector spaces. If $(x_n)_{n \in \mathbb{N}}$ is a sequence in a normed vector space E and there is an element $l \in E$ such that $\lim_{n\to\infty} \|x_n - l\| = 0$, then we say that the sequence is *convergent*. It is easy to see that the element l must be unique. We call l the *limit* of the sequence and write $\lim_{n\to\infty} x_n = l$. We will in general abbreviate $(x_n)_{n \in \mathbb{N}}$ to (x_n) and $\lim_{n\to\infty} x_n = l$ to $\lim x_n = l$. The following result is elementary.

Proposition 1.2. *If (x_n) and (y_n) are convergent sequences in E, with $\lim x_n = l_1$ and $\lim y_n = l_2$, and $\lambda \in \mathbb{R}$, then*

$$\lim(x_n + y_n) = l_1 + l_2 \qquad \text{and} \qquad \lim(\lambda x_n) = \lambda l_1.$$

Suppose now that we have two normed vector spaces, $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$. Let A be a subset of E, f a mapping of A into F and $a \in A$. We say that f is *continuous at a* if the following condition is satisfied:

for all $\epsilon>0$, there exists $\delta>0$ such that, if $x \in A$ and $\|x-a\|_E<\delta$,

then $\|f(x)-f(a)\|<\epsilon$.

If f is continuous at every point $a \in A$, then we say that f is *continuous* (on A). Finally, if $A \subset E$ and $B \subset F$ and $f : A \longrightarrow B$ is a continuous bijection such that the inverse mapping f^{-1} is also continuous, then we say that f is a *homeomorphism*.

Exercise 1.7. Suppose that $\|\cdot\|_E$ and $\|\cdot\|_E^\times$ are equivalent norms on E and $\|\cdot\|_F$ and $\|\cdot\|_F^\times$ equivalent norms on F. Show that f is continuous at a (resp. continuous) for the pair $(\|\cdot\|_E, \|\cdot\|_F)$ if and only if f is continuous at a (resp. continuous) for the pair $(\|\cdot\|_E^\times, \|\cdot\|_F^\times)$.

Proposition 1.3. *The norm on a normed vector space is a continuous function.*

Proof. We have

$$\|x\| = \|x - y + y\| \le \|x - y\| + \|y\| \Longrightarrow \|x\| - \|y\| \le \|x - y\|.$$

In the same way $\|y\| - \|x\| \le \|y - x\|$. As $\|y - x\| = \|x - y\|$, we have

$$|\|x\| - \|y\|| \le \|x - y\|,$$

and hence the continuity. □

The next result is also elementary.

Proposition 1.4. *Let E and F be normed vector spaces, $A \subset E$, $a \in A$, f and g mappings from E into F and $\lambda \in \mathbb{R}$.*

- *If f and g are continuous at a, then so is $f + g$.*
- *If f is continuous at a, then so is λf.*
- *If α is a real-valued function defined on E and both f and α are continuous at a, then so is αf.*

Corollary 1.1. *The functions $f : E \longrightarrow F$ which are continuous at a (resp. continuous) form a vector space.*

We now consider cartesian products of normed vector spaces. Let $(E_1, \|\cdot\|_{E_1}), \ldots, (E_p, \|\cdot\|_{E_p})$ be normed vector spaces. The cartesian product $E_1 \times \cdots \times E_p$ is a vector space. For $(x_1, \ldots, x_p) \in E_1 \times \cdots \times E_p$ we set

$$\|(x_1, \ldots, x_p)\|_S = \|x_1\|_{E_1} + \cdots + \|x_p\|_{E_p} \quad \text{and}$$
$$\|(x_1, \ldots, x_p)\|_M = \max(\|x_1\|_{E_1}, \ldots, \|x_p\|_{E_p}).$$

There is no difficulty in seeing that $\|\cdot\|_S$ and $\|\cdot\|_M$ are equivalent norms on $E_1 \times \cdots \times E_p$. In general, we will use the second norm, which we will refer to as the usual norm. If $E_1 = \cdots = E_p = \mathbb{R}$ and $\|\cdot\|_{E_i} = |\cdot|$ for all i, then $\|\cdot\|_S = \|\cdot\|_1$ and $\|\cdot\|_M = \|\cdot\|_\infty$.

Proposition 1.5. *Let $(E, \|\cdot\|)$ be a normed vector space.*

- *The mapping $f : E \times E \longrightarrow E, (x, y) \longmapsto x + y$ is continuous.*
- *The mapping $g : \mathbb{R} \times E \longrightarrow E, (\lambda, x) \longmapsto \lambda x$ is continuous.*

Proof. Let us first consider f. We have

$$\begin{aligned}\|(x, y) - (a, b)\|_M < \epsilon &\Longrightarrow \|x - a\| < \epsilon, \ \|y - b\| < \epsilon \\ &\Longrightarrow \|(x + y) - (a + b)\| \le \|x - a\| + \|y - b\| < 2\epsilon,\end{aligned}$$

hence f is continuous at (a, b).

We now consider g. If $\|(\lambda, x) - (\alpha, a)\| < \epsilon$, then $|\lambda - \alpha| < \epsilon$ and $\|x - a\| < \epsilon$ and so

$$\|\lambda x - \alpha a\| = \|\lambda x - \lambda a + \lambda a - \alpha a\| \le |\lambda|\|x - a\| + |\lambda - \alpha|\|a\| < (|\alpha| + \epsilon)\epsilon + \epsilon\|a\|,$$

therefore g is continuous at (α, a). □

A composition of real-valued continuous functions of a real variable is continuous. We have a generalization of this result.

Proposition 1.6. *Let E, F and G be normed vector spaces, $A \subset E$, $B \subset F$, f a mapping of A into F and g a mapping of B into G. If $f(A) \subset B$, f is continuous at $a \in A$ and g continuous at $f(a)$, then $g \circ f$ is continuous at a.*

Proof. Let us take $\epsilon > 0$. As g is continuous at $f(a)$, there exists $\delta > 0$ such that, if $y \in B$ and $\|y - f(a)\|_F < \delta$, then $\|g(y) - g(f(a))\|_G < \epsilon$. As f is continuous at a, there exists $\alpha > 0$ such that, if $x \in A$ and $\|x - a\|_E < \alpha$, then $\|f(x) - f(a)\|_F < \delta$. This implies that $\|g(f(x)) - g(f(a))\|_G < \epsilon$. Therefore $g \circ f$ is continuous at a. □

Corollary 1.2. *If $A \subset E$ and $f : A \longrightarrow \mathbb{R}$ is continuous and nonzero on A, then the function $g = \frac{1}{f}$ is continuous on A.*